DISCUSSION

Poll\textsuperscript{1}: Are you a Late Merger or Early Merger? Updated Result: Apparently most students who attend discussion are late mergers.

Introduction to Traffic Modeling: What we wish to study is an idealized unidirectional single lane road with vehicles that drive according to some rational law, which may be introduced during the process of modeling. When no cars are allowed to pass one another, the problem greatly simplifies and becomes a more tractable topic to be covered in 3 weeks. One may immediately raise the question, "Why study this problem if it nearly never arises in the actual world?" A perfectly valid point. One reason is because the situation that arises mathematically is not only relevant to this specific traffic problem, but also in contexts such as 1D fluid flows, which are very prevalent in the real world. A few examples could be methane gas flowing through a carbon nanotube or blood flowing through a capillary. Furthermore, if one wishes to study 2D or 3D fluid flows, it is instructive to learn the 1D case as a foundation for one’s understanding.

What do we need to begin to talk about traffic flow? First, we will need to specify the variables ρ(\(x,t\)), q(\(x,t\)), and u(\(x,t\)). On any 2\(\Delta x\) length segment of road, we could define the density of cars to be

\[
ρ(x_0,t) = \frac{\text{Cars in the interval } [x_0-\Delta x,x_0+\Delta x]}{2\Delta x}.
\] (1)

In the same vein, we can define the flux of cars (or in the book, the flow of cars) to be the number of cars passing through a point over a given time interval,

\[
q(x_0,t_0) = \frac{\text{Cars passing through } x_0 \text{ during the interval } [t_0-\Delta t,t_0+\Delta t]}{2\Delta t}.
\] (2)

Last, the velocity field describing the speed of the cars at position \(x_0\) at time \(t_0\) is denoted by \(u(x_0,t_0)\). The goal is to treat traffic mostly like a continuous 1D fluid. In order for this to be a reasonable approximation, we must look at situations where there are sufficiently many cars on the road. This assumption therefore becomes more valid as we either increase our length scale (looking at segments of road much much bigger than the length of a car), or in situations where there are many cars packed into a small segment.

If we are to plot \(ρ(x,t)\) at a given time \(t_0\) as it is currently defined, we would get a discontinuous graph, with a jump discontinuity every time we include another car in our 2\(\Delta x\) interval. Because we wish to treat the traffic as a fluid-like object, we will utilize the so-called continuum approximation somewhat discussed above. This assumption that there is a continuum of cars means that \(ρ,q,u\) will be smoothly varying functions, and so we can take the limit of \(\Delta t,\Delta x\to 0\) and still have a useful and meaningful understanding of what it means to have a car density or flux at a point \((x,t)\). More of this will be discussed in section.

Drawing an analogy to fluids, we expect that some sort of conservation of mass law should exist. This will provide us with a governing equation for how the density \(ρ\) will evolve in time and space. If we look at the region \([x_0-\Delta x,x_0+\Delta x] \times [t_0-\Delta t,t_0+\Delta t]\), we should require that \{The number of cars in the spatial interval at \(t_0+\Delta t\}\} - \{The number of cars in the spatial interval at \(t_0-\Delta t\}\} = \{The net number of cars entering the spatial interval through \(x_0-\Delta x\) during the time interval\} - \{The net number of cars leaving

\textsuperscript{1}A poll for the class just for fun. There are usually two camps when discussing how to exit the freeway. The first camp of late-mergers creep all the way up to the exit and merge at the last minute, and the second camp consists of those who sit and wait patiently in the long line to exit.
the spatial interval through \( x_0 + \Delta x \) during the time interval \( \Delta t \). Writing this in terms of density and flux, one has,

\[
\rho(x_0, t_0 + \Delta t) - \rho(x_0, t_0 - \Delta t) = \frac{q(x_0 - \Delta x, t_0) - q(x_0 + \Delta x, t_0)}{2\Delta x}.
\]

(3)

Due to the continuum hypothesis that \( \rho \) and \( q \) vary smoothly, we can use a Taylor’s approximation to write at \( (x_0, t_0) \),

\[
\frac{\rho + \Delta t \rho_t + \frac{\Delta x^2}{2} \rho_{tt} + \cdots - \rho + \Delta t \rho_t - \frac{\Delta x^2}{2} \rho_{tt} + \cdots}{2\Delta t} = \frac{q - \Delta x q_x + \frac{\Delta x^2}{2} q_{xx} + \cdots - q - \Delta x q_x - \frac{\Delta x^2}{2} q_{xx} + \cdots}{2\Delta x}.
\]

(4)

One can now take the limit as \( \Delta t, \Delta x \to 0 \) to get

\[
\rho_t = -q_x,
\]

(5)

or

\[
\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0
\]

(6)

This equation has many names because it arises in many different fields and problems. Here, I will probably refer to it as the mass conservation law, or the advection equation. This is our first instance of a partial differential equation, meaning we have \( \rho \) varying over time and space and we have that its value at any point \( (x, t) \) depends on the balance of the flux spatial derivative and the density’s time derivative. One more simplification can be added by looking at the units of flux and making the reasonable assumption\(^2\) that \( q = u \rho \). Upon introducing this, we have the PDE,

\[
\rho_t + (\rho u)_x = 0.
\]

(7)

The modeling problem that arises here is that in order for us to find the density of cars, we need to know what the velocity field of the cars is over the relevant segment of road. Here is where the richness in modeling traffic comes in. Many scientists in the past have taken measurements of the speed of cars versus the density of cars on the road, i.e. \( u = u(\rho(x, t)) \). These empirical functions which one gets from fitting a function to the measured data are then used as so-called constitutive laws, which make the PDE above solvable\(^3\).

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\(^2\)This is immediate when the density distribution is uniform, and can be explicitly shown for many other situations. However, there doesn’t exist any rigorous proof of this, so many authors just take this to be the definition of \( q \).

\(^3\)In certain situations, a nonlinear 1st order PDE is not solvable at all, or only solvable for a brief time period. This is due to the formation of shock waves in the solution, which will be discussed at a later date.