HOMEWORK 9: SOLUTIONS/HINTS

(32.1) Let $\mathcal{P} = \{0 = t_0 < t_1 < \ldots < t_n = b\}$ be a partition of $[0, b]$, then

$$\begin{aligned}
U(f,\mathcal{P}) &= \sum_{i=1}^{n} \sup(x^3, [t_{i-1}, t_i]) \cdot (t_i - t_{i-1}) \\
&= \sum_{i=1}^{n} t_i^3 \cdot (t_i - t_{i-1}) \\
L(f,\mathcal{P}) &= \sum_{i=1}^{n} \inf(x^3, [t_{i-1}, t_i]) \cdot (t_i - t_{i-1}) \\
&= \sum_{i=1}^{n} t_{i-1}^3 \cdot (t_i - t_{i-1}).
\end{aligned}$$

For $t_i = \frac{i \cdot b}{n}$, we then get

$$\begin{aligned}
U(f,\mathcal{P}) &= \frac{b^4}{n^4} \sum_{i=1}^{n} i^3 = \frac{b^4}{n^4} \left(\sum_{i=1}^{n} i \right)^2 \\
&= \frac{b^4}{n^4} \left(\frac{n \cdot (n+1)}{2} \right)^2 = \frac{b^4}{4} \left(\frac{n+1}{n} \right)^2 \quad \text{and} \\
L(f,\mathcal{P}) &= \frac{b^4}{n^4} \sum_{i=1}^{n} (i-1)^3 = \frac{b^4}{n^4} \sum_{i=1}^{n-1} i^3 = \frac{b^4}{n^4} \left(\sum_{i=1}^{n-1} i \right)^2 \\
&= \frac{b^4}{n^4} \left(\frac{n \cdot (n-1)}{2} \right)^2 = \frac{b^4}{4} \left(\frac{n-1}{n} \right)^2.
\end{aligned}$$

Letting $n \to \infty$ we get that $U(f) = \frac{b^4}{4} = L(f)$.

(32.2) Note that $\sup(f, [t_{i-1}, t_i]) = t_i$ and $\inf(f, [t_{i-1}, t_i]) = 0$ for any $t_{i-1} < t_i$ since the rational respectively irrational numbers are dense. So we get for any partition

$$\mathcal{P} := \{0 = t_0 < t_1 < \ldots < t_n = b\}$$

that $L(f,\mathcal{P}) = 0$ and $U(f,\mathcal{P}) = \sum_{i=1}^{n} t_i(t_i - t_{i-1})$. In particular we get for $t_i = \frac{i \cdot b}{n}$, that

$$U(f,\mathcal{P}) = \frac{b^2}{n^2} \sum_{i}^{n} i = \frac{b^2}{2} \cdot \frac{n(n+1)}{n^2}$$

and hence by letting $n \to \infty$ that

$$U(f) = \frac{b^2}{2} \neq 0 = L(f)$$

for $b > 0$ and thus f is not integrable.

(32.4) Let $\mathcal{P} \subset \mathcal{Q}$ be two partitions of $[a, b]$. We prove that $L(f, \mathcal{P}) \leq L(f, \mathcal{Q}) \leq U(f, \mathcal{Q}) \leq U(f, \mathcal{P})$ by induction on $n := \#\mathcal{Q} - \#\mathcal{P}$. The base case $n = 1$ is done in the proof of Lemma 32.2. For $n \geq 2$, let $\mathcal{R}$ be another partition of $[a, b]$ such that $\mathcal{P} \subsetneq \mathcal{R} \subsetneq \mathcal{Q}$, then $\#\mathcal{Q} - \#\mathcal{R}$ and $\#\mathcal{R} - \#\mathcal{P}$ are both less than n and so we have by induction assumption that

$$L(f, \mathcal{P}) \leq L(f, \mathcal{R}) \leq L(f, \mathcal{Q}) \leq U(f, \mathcal{Q}) \leq U(f, \mathcal{R}) \leq U(f, \mathcal{P}).$$

(33.4) Let

$$f(x) := \begin{cases} 1: & x \text{ rational} \\ -1: & x \text{ irrational} \end{cases}$$

for $x \in [0, 1]$, then $\sup(f, [t_{i-1}, t_i]) = 1$ and $\inf(f, [t_{i-1}, t_i]) = -1$ for any $t_{i-1} < t_i$ and therefore

$$U(f) = 1 \neq -1 = L(f).$$

So f is not integrable, however $|f| = 1$ is obviously integrable.

(33.7) Let $S \subset [a, b]$ and $x_0, y_0 \in \mathcal{S}$, then

$$f^2(x_0) - f^2(y_0) = (f(x_0) - f(y_0)) \cdot (f(x_0) + f(y_0)) \leq 2B(f(x_0) - f(y_0)).$$

It then follows that

$$\sup(f^2, S) - \inf(f^2, S) \leq 2B(\sup(f, S) - \inf(f, S)),$$

so we then get for any partition $\mathcal{P} = \{a = t_0 < t_1 < \ldots < t_n = b\}$ of $[a, b]$ that

$$\begin{aligned} & U(f^2, \mathcal{P}) - L(f^2, \mathcal{P}) \\ = \; & \sum_{i=1}^{n} \big(\sup(f^2, [t_{i-1}, t_i]) - \inf(f^2, [t_{i-1}, t_i])\big) \cdot (t_i - t_{i-1}) \\ \leq \; & 2B \sum_{i=1}^{n} (\sup(f, [t_{i-1}, t_i]) - \inf(f, [t_{i-1}, t_i])) \cdot (t_i - t_{i-1}) \\ = \; & 2B \cdot (U(f, \mathcal{P}) - L(f, \mathcal{P})). \end{aligned}$$

So we get that

$$U(f^2) - L(f^2) \leq 2B \cdot (U(f) - L(f)).$$

Since f is integrable we get that $L(f) = U(f)$ and hence $L(f^2) = U(f^2)$ by the above inequality and thus f^2 is integrable.