# An example of the difference between weak closedness and sequential weak closedness for Banach spaces

In response to a query in section, here I'll give hints on the analysis of an example of a Banach space $X$ and a subset $A$ of $X$ whose weak closure and weak sequential closure are different.

In fact, the $X$ we choose is very familiar: $X := \ell^1_{\mathbb{R}}(S) = \ell^1$ for some countably infinite set $S$ (it doesn't matter which such $S$ we choose). In this space we will let $A$ be the set of those functions $f : S \to \mathbb{R}$ for which there are $s, t \in S$ such that $f(s) = 1$, $f(t) = -1$ and $f(z) = 0$ for all $z \in S \setminus \{s, t\}$ (note, in particular, that this requires $s \neq t$). Let us denote this particular $f$ by $\delta_s - \delta_t$.

**Claim 1** *The zero function $0 \in \ell^1$ lies in the weak closure of $A$, but not in its sequential weak closure.*

**Proof** This falls into two parts. I'll here give a complete proof for the first part, and reduce the second to an exercise for the reader.

**Part 1** We show that 0 lies in the weak closure of $A$. We use the identification of $(\ell^1)^*$ with $\ell^\infty$ (Theorem 6.15 in Folland's *Real Analysis*), so we need to show that for any $g_1, g_2, \ldots, g_k \in \ell^\infty$ and $\varepsilon > 0$ there is a function $f \in A$ with $|\langle g_i, f\rangle| < \varepsilon$ for all $i \leq k$.

To do this, first observe that every $g_i$ is bounded, so the combined function

$$\mathbf{g} : \mathbb{N} \to \mathbb{R}^k : x \mapsto (g_1(x), g_2(x), \ldots, g_k(x))$$

takes values in some bounded subset $K \subseteq \mathbb{R}^k$. It follows that $K$ is totally bounded, since this is equivalent to boundedness for subsets of $\mathbb{R}^k$. Therefore we can cover $K$ by finitely many open balls of radius $\varepsilon/2$ for the norm $\|\cdot\|_\infty$

on $\mathbb{R}^k$, and so by cutting these balls down further we can partition $K$ into finitely many subsets $K_1$, $K_2$, $\dots$, $K_N$ each of radius less than $\varepsilon$.

Now the pulled-back sets $\mathbf{g}^{-1}(K_1), \mathbf{g}^{-1}(K_2), \dots, \mathbf{g}^{-1}(K_N)$ form a partition of the infinite set $S$, and so one of these sets must be infinite, say $\mathbf{g}^{-1}(K_i)$. Therefore we can pick distinct $s, t \in \mathbf{g}^{-1}(K_i)$, and now set $f := \delta_s - \delta_t$. Now we have that

$$|\langle g_i, f\rangle| = |g_i(s) - g_i(t)|,$$

but we know that $\mathbf{g}(s)$ and $\mathbf{g}(t)$ both lie in $K_i$, and hence

$$|g_i(s) - g_i(t)| \leq \|\mathbf{g}(s) - \mathbf{g}(t)\|_\infty \leq \operatorname{diam} K_i < \varepsilon,$$

so $|\langle g_i, f\rangle| < \varepsilon$, as desired.

**Part 2** On the other hand, suppose now that $f_n$ is a sequence in $A$, say $f_n = \delta_{s_n} - \delta_{t_n}$. We will show that this cannot converge weakly to $0 \in X$: that is, that there is a subsequence $(f_{n_k})_{k\geq 1}$ and some $g \in \ell^\infty$ such that $\langle g, f_{n_k}\rangle$ stays bounded away from $\langle g, 0\rangle = 0$ as $k \to \infty$.

In fact we will take $g$ of the form $1_E$ for some carefully-chosen $E \subseteq \mathbb{N}$. In this case we have $\langle g, f_n\rangle = |E \cap \{s_n\}| - |E \cap \{t_n\}|$, so the desired result is now a consequence of the following exercise:

**Exercise** Suppose that $(s_n, t_n)_{n\geq 1}$ is a sequence of pairs in $S \times S$ with $s_n \neq t_n$ for all $n \geq 1$. As a matter of notation, let us say that a sequence $s_n$ in $S$ *converges to* $\infty$, or write that $s_n \to \infty$, if for every finite subset $F \subset S$ there is an $N$ such that $s_n \in S \setminus F$ for all $n \geq N$.

**(a)** Show that at least one of the following must be true:

- Some pair $(s, t)$ occurs in the sequence infinitely many times, and therefore there is some $s$ with [$s_n = s$ but $t_n \neq s$] for infinitely many $n$;
- There is an infinite subsequence $(s_{n_k}, t_{n_k})_{k\geq 1}$ such that $s_{n_k} \to \infty$ as $k \to \infty$ but $t_{n_k}$ is constant;
- There is an infinite subsequence $(s_{n_k}, t_{n_k})_{k\geq 1}$ such that $t_{n_k} \to \infty$ as $k \to \infty$ but $s_{n_k}$ is constant;
- Both $s_n \to \infty$ and $t_n \to \infty$.

**(b)** Deduce that for any such sequence of pairs there is a subset $E \subseteq S$ with $[E \ni s_n$ but $E \not\ni t_n]$ for infinitely many $n$, so that $\langle 1_E, f_n \rangle = 1$ for some subsequence of $f_n$.

By the above remarks this completes the proof of Claim 1. $\square$

**Remark** Notice that in this example the set $A$ is actually countable, so this feature by itself is not enough to close the gap between closedness and sequential closedness.

**Corollary 1** *The weak sequential closure of* $A$ *(that is, the set of all weak limit points of weakly convergent sequences in* $A$*) is not the same as its weak closure (the set of all weak limit points of weakly convergent nets in* $A$*), since the latter contains* $0$ *but the former does not.*

**Follow-up exercise** Describe either the weak closure or weak sequential closure of $A$ as explicitly as you can.