A COUNTEREXAMPLE IN VON NEUMANN ALGEBRA DYNAMICS

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ABSTRACT. In this essay we construct a noncommutative von Neumann dynamical system with certain pathological properties from the viewpoint of analogies with classical ergodic theory. These relate to obstructions to the existence of special kinds of subsystem or subextension. We will build our example as an extension of a commutative dynamical system via a unitary cocycle acting on a Hilbert bundle, and will see that the well-known possibility of pathological behaviour for such unitary cocycles can, with a little care, be translated into pathological behaviour for von Neumann dynamical system extensions.

CONTENTS

1. INTRODUCTION

We study here a particular von Neumann dynamical system: an ordered triple $(\mathfrak{M}, \theta, \alpha)$ consisting of a von Neumann algebra $\mathfrak{M}$ equipped with a faithful normal state θ (serving as the analog of the measure) and a θ-preserving automorphism α. In particular, we construct an example of such a system, naturally represented on a particular Hilbert space such that the automorphism is given by some unitary on that space, which exhibits various 'strictly noncommutative' behaviour, in that it violates the obvious noncommutative analogs of certain standard results of ergodic theory for probability-preserving systems. Our example is obtained as an extension of a classical (that is, commutative) measurable dynamical system via a carefully-chosen Hilbert bundle and unitary cocycle over that system; the 'essentially noncommutative' behaviour of this extension will follow from that of the cocycle.

After specifying our example, we will consider the possibility that the demand of global invariance under an automorphism of a von Neumann algebra can obstruct the extensions of a subalgebra. Suppose we are given a von Neumann dynamical system $(\mathfrak{M},\theta,\alpha)$ spatially represented on some Hilbert space $\mathfrak{H}$, and suppose further that $\mathfrak{M}$ contains some α-invariant Abelian subalgebra $\mathfrak{N}$. Is it possible that $\mathfrak{N}$ is not maximal among all Abelian subalgebras of $\mathfrak{M}$, but is maximal among those that are α-invariant? We will show that the answer is Yes; in doing so, we will obtain a bound on the applicability of a certain model of Hilbert space operator.

2. The main construction

2.1. **Hilbert bundles, cocyles and tensor products.** Let (Ω,Σ,μ,τ) be a Lebesgue probability-preserving system and $\mathfrak{H}_0$ some fixed complex Hilbert space (which we will later take finite-dimensional). Henceforth, we will let $\mathfrak{M}$ be the algebra $L^\infty_{\mathcal{B}(\mathfrak{H}_0)}(\mu)$ of bounded measurable functions from Ω to $\mathcal{B}(\mathfrak{H}_0)$, defined up to μ-almost everywhere equality. We will refer to these as **operator cocycles**. As is standard, $\mathfrak{M}$ can be represented through its natural pointwise action on $\mathfrak{H} := L^2_{\mathfrak{H}_0}(\mu)$; given $A\in\mathfrak{M}$ we will sometimes write $\mathrm{Mult}_A\in\mathcal{B}(\mathfrak{H})$ for this operator. In this picture we can identify $L^\infty_{\mathbb{C}}(\mu)$ as the centre of $\mathfrak{M}$ and write $\mathfrak{M}$ as the direct integral in the obvious way:

$$\mathfrak{M} = \int_\Omega^\oplus \mathcal{B}(\mathfrak{H}_\omega)\,\mu(\mathrm{d}\omega),$$

where we write $\mathfrak{H}_\omega$ for a copy of $\mathfrak{H}_0$ associated with the point $\omega\in\Omega$. In addition, $\mathfrak{M}$ also carries the invariant faithful normal trace θ given by

$$\theta(A) = \int_\Omega \mathrm{Tr}(A(\omega))\,\mu(\mathrm{d}\omega)$$

for $A:\Omega\to\mathcal{B}(\mathfrak{H}_0)$ in $\mathfrak{M}$.

The unitary members of $\mathfrak{M}$ correspond precisely to **unitary cocycles**: measurable maps $\Phi:\Omega\to\mathcal{U}(\mathfrak{H}_0)$ to the unitary group of $\mathfrak{H}_0$. Given such a Φ let us also define the unitary operator $U^{\tau,\Phi}$ on $\mathfrak{H}$ by

$$U^{\tau,\Phi}f(\omega) = \Phi(\tau(\omega))f(\tau(\omega)) = \big((\mathrm{Mult}_\Phi\circ\tau^\#)f\big)(\omega),$$

writing $\tau^\#$ for the unitary operator on $\mathfrak{H}$ given by composition with τ. We will refer to a unitary $U^{\tau,\Phi}$ constructed in this way as a **twisted composition**

operator over τ. This definition can clearly be extended to any operator of the form $\mathrm{Mult}_A \circ \tau^{\#}$ for some operator cocycle $A \in \mathfrak{M}$; we will refer to these as **generalized weighted composition operators over** τ.

Given a unitary $U \in \mathcal{U}(\mathfrak{H})$ for any Hilbert space $\mathfrak{H}$ and a von Neumann algebra $\mathfrak{M}$ on $\mathfrak{H}$ with $U\mathfrak{M}U^* = \mathfrak{M}$, we will write α^U for the automorphism on $\mathfrak{M}$ defined by conjugation by U; for the twisted composition unitaries $U^{\tau,\Phi}$ constructed above we will abbreviate $\alpha^{U^{\tau,\Phi}}$ to $\alpha^{\tau,\Phi}$.

2.2. **The construction of the example.** We now specialize to the particular dynamical systems and cocycles that we will use later. We take the fibre space $\mathfrak{H}_0$ to be $\mathbb{C}^2$ (although the arguments that follow could be adapted to any space of finite dimension greater than 1) and so $\mathcal{U}(\mathfrak{H}_0) = \mathrm{U}(2)$.

For our underlying probability-preserving system $(\Omega, \Sigma, \mu, \tau)$ we take the $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$-Bernoulli shift system given by the coordinate right-shift on $\{1, 2, 3, 4\}^{\mathbb{Z}}$ with its usual product measure space structure. (We will see below that there is a related construction over the simpler $(\frac{1}{2}, \frac{1}{2})$-Bernoulli shift, but at a certain point this then necessitates a slightly longer argument.) Write $\pi_i : \Omega \to \{1, 2, 3, 4\}$ for the projection onto the i^{th} coordinate. Our unitary cocycles Φ will depend only on the zeroth coordinate in Ω, and so take at most four distinct values. Moreover, these will be of the form V_1, V_2, $V_3 := V_1^{-1}$ and $V_4 := V_2^{-1}$, so that $\Phi(\omega) = V_{\pi_0(\omega)}$. It is so that we can introduce two distinct unitaries and also their inverses that we work over a shift alphabet of size four.

We will not specify particular V_1 and V_2 in $\mathrm{U}(2)$, but will show that a uniform random pair is almost sure to have the properties we will call on later. This relies on a manipulation of the dynamics of these cocycles through the following lemma.

Lemma 2.1. *Let V_1, $V_2 \in \mathrm{U}(2)$. Consider the set W_n of all words of length N in the free group F_2 on two generators, and, given such a word w, let $\mathbf{w}_{V_1,V_2}(w)$ be the corresponding word in V_1, V_2, V_1^{-1} and V_2^{-1}. Then for the usual Haar measure on $\mathrm{U}(2) \times \mathrm{U}(2)$, almost every pair (V_1, V_2) is such that the empirical measures*

$$\frac{1}{|W_n|} \sum_{w \in W_n} \delta_{\mathbf{w}_{V_1,V_2}(w)}$$

converge vaguely to the Haar measure on $\mathrm{U}(2)$ as $n \to \infty$.

Remark The proof we give below relies on a much more abstract result about ergodic actions of free groups: the Nevo-Stein ergodic theorem ([3]). However, it might be interesting to seek a more elementary approach to this special case, and such an elementary approach might then also remove the need to include the inverses V_1^{-1}, V_2^{-1}. $\lhd$

Proof We can identify the words $\mathbf{w}_{V_1,V_2}(w)$ as the images of $\mathbf{1}_{\mathbb{C}^2}$ under the action of F_2 on $\mathrm{U}(2)$ for which the two generators are sent to left-multiplication by V_1 and V_2 respectively. It is now routine to verify (for example, by reducing the question to a consideration of the irreducible representations of $\mathrm{U}(2)$ through an appeal to the Peter-Weyl Theorem) that this action is ergodic for almost every choice of (V_1, V_2); and now the Nevo-Stein free group ergodic theorem (see [3]) gives pointwise convergence of the ergodic averages corresponding to the points $\mathbf{w}_{V_1,V_2}(w)$, $w \in W_n$, and so also the desired vague convergence of the empirical measures. (Note that in the first instance the Nevo-Stein Theorem gives this convergence only for the sets W_n corresponding to even lengths n; however, it is clear that in our case there is enough rigidity in the action that this implies the full result.) $\square$

Let us now write out explicitly the form of the images $(\alpha^{\tau,\Phi})^n(A)$ of A for an operator-valued cocycle $A : \Omega \to \mathcal{B}(\mathfrak{H}_0)$:

$$\begin{aligned}(\alpha^{\tau,\Phi})^n A(\omega) &= \Phi(\tau(\omega))\Phi(\tau^2(\omega))\cdots\Phi(\tau^n(\omega))A(\tau^n(\omega))\Phi(\tau^n(\omega))^*\cdots\Phi(\tau^2(\omega))^*\Phi(\tau(\omega))^* \\ &= V_{\omega_1}V_{\omega_2}\cdots V_{\omega_n}A(\tau^n(\omega))V_{\omega_n}^*\cdots V_{\omega_2}^*V_{\omega_1}^*.\end{aligned}$$

From this, we can deduce the particular property that we will need for our counterexamples.

Lemma 2.2. *Suppose that $k \geq 0$ and $\mathcal{A}$ is the algebra of all clopen subsets of Ω depending only on the coordinates $-k, -k+1, \ldots, k$, and that $A : \Omega \to \mathrm{Mat}_{2\times 2}(\mathbb{C})$ is a measurable operator-valued cocycle that is actually $\mathcal{A}$-measurable. Suppose also that $C \subseteq \Omega$ is a finite dimensional cylinder of the form $\{\omega : \pi_i(\omega) = \eta_i \text{ for } -m \leq i \leq m\}$. Then as $n \to \infty$ the conditional distribution of $(\alpha^{\tau,\Phi})^n(A)$ inside the set C converges vaguely to the combined distribution of a matrix sampled from the distribution of A conjugated by an independent uniformly distributed unitary in* $\mathrm{U}(2)$.

Proof This is a straightforward consequence of our above calculation. By enlarging either m or k as necessary, we may assume they are equal (for if the conclusion holds for each of a partition of C into cylinder sets depending on more coordinates, then it certainly holds for C itself). If $n \geq 2k+1$ then $(\alpha^{\tau,\Phi})^n(A)$ is given by

$$V_{\omega_1}V_{\omega_2}\cdots V_{\omega_n}A(\omega_{n-k},\omega_{n+1-k},\ldots,\omega_{n+k})V_{\omega_1}^*V_{\omega_2}^*\cdots V_{\omega_n}^*.$$

We may re-write this as

$$W_{n,1}W_{n,2}W_{n,3}A(\omega|_{\{n-k,n-k+1,\ldots,n+k\}})W_{n,3}^*W_{n,2}^*W_{n,1}^*,$$

where

$$\begin{aligned} W_{n,1} &:= V_{\omega_1}V_{\omega_2}\cdots V_{\omega_k} \\ W_{n,2} &:= V_{\omega_{k+1}}V_{\omega_{k+2}}\cdots V_{\omega_{n-k-1}}, \\ W_{n,3} &:= V_{\omega_{n-k}}V_{\omega_{n-k+1}}\cdots V_{\omega_n}. \end{aligned}$$

Now, as $n \to \infty$, by Lemma 2.1 the unitary $W_{n,2}$ converge vaguely in distribution to the Haar measure on $\mathrm{U}(2)$ (for their distribution is clearly precisely that of the images of words of length $n - 2k$ under the action of F_2 considered in Lemma 2.1). Thus, the distribution of $(\alpha^{\tau,\Phi})^n(A)$ conditioned onto the cylinder C, where $\omega_i = \eta_i$ is fixed for $-k \leq i \leq k$, is the mixture of the conjugation-images of the values $W_{n,3}A(\omega|_{\{n-k,n-k+1,\ldots,n+k\}})W_{n,3}^*$ under the $W_{n,2}$ whose distributions are close to uniform. Since the words $W_{n,3}$ in V_1, V_2 and their inverses are of length k, independent of n, they can take only finitely many possible values; the same reasoning applies to $W_{n,1}$. Thus our overall conditional distribution in C is a mixture of separate distributions, each of which is a value of A on a single cylinder set of $\mathcal{A}$, conjugated by one of finitely many possible unitaries, and then conjugated by a random unitary of distribution increasingly close (in the vague topology) to uniform, and then conjugated by another of finitely many possible unitaries. Since the left- or right-product of a uniformly distributed random unitary by any given unitary is still uniformly distributed, these random conjugates of values of A converge (vaguely in distribution) to a uniformly random member of their unitary conjugacy class; recombining these as a mixture gives the claimed overall distribution. □

Remark It seems likely that a slightly more careful analysis would permit a natural extension of this result to finitely-valued unitary cocycles over some more general strongly mixing (or perhaps just weakly mixing) measure-preserving transformations. The main extra difficulty lies in the use of a result analogous to Lemma 2.1: if our underlying transformation is strongly mixing but not actually Bernoulli with respect to the partition generated by the unitary cocycle, then the distribution of the different possible words in V_1, V_2 and their inverses given by the $W_{n,2}$ above is no longer uniform over these words, and so no longer corresponds to the empirical distribution of a rotation action of F_2 on $\mathrm{U}(2)$. Thus we would need to replace Lemma 2.1 with a more general result about the limiting distributions of long products of randomly-chosen unitaries that do not commute and are only quite weakly dependent one on another; while such a result seems intuitively plausible, it is not so simple as Lemma 2.1 and we will not consider it further here. ◁

3. APPLICATION: EXTENDING GLOBALLY-FIXED ABELIAN SUBALGEBRAS

The Spectral Theorem amounts to a simple necessary and sufficient condition (that of normality) for a bounded operator T on a separable Hilbert space to be spatially isomorphic to a multiplication operator on $L^2_{\mathbb{C}}(\mu)$ for some Lebesgue probability space (Ω, Σ, μ). In addition, the normality of T can be determined from its polar decomposition: if $T = |T|U$ for a positive $|T|$ and unitary U, then T is normal if and only if $|T|$ and U commute.

One possible extension of this would be to ask when T (which we will henceforth assume invertible, for simplicity) is spatially isomorphic to a generalized weighted composition operator over τ on $L^2_{\mathbb{C}}(\mu)$ for some nonsingular measurable system $(\Omega, \Sigma, \mu, \tau)$ (note that we clearly cannot now expect τ to be probability-preserving). In this case we can write our generalized weighted composition as $\mathrm{Mult}_h \circ \mathrm{Mult}_\phi \circ \tau^\#$ for some non-negative $h \in L^\infty_{\mathbb{C}}(\mu)$ and some measurable $\phi : \Omega \to \mathbb{T}$; we see at once that this operator has polar decomposition with positive part Mult_h and unitary part $\mathrm{Mult}_\phi \circ \tau^\#$, but now these parts need not commute (indeed, they will not unless $h \cdot \phi$ is τ-invariant). However, it is still true that any conjugate of a multiplication operator on $L^2_{\mathbb{C}}(\mu)$ with underlying function $g \in L^\infty_{\mathbb{C}}(\mu)$ by the n^{th} power of the unitary part $\mathrm{Mult}_\phi \circ \tau^\#$ must be just the multiplication operator corresponding to $g \circ \tau^n$; therefore, in particular, these conjugates all commute, and so a necessary condition that an invertible T be representable this way is that its positive part commute with all of its conjugates under the unitary part. We will use our previously constructed action to show that this, however, is not sufficient.

Indeed, this condition asserts precisely that if $T = |T|U$ then conjugation by U, regarded as an inner automorphism of the full operator algebra $\mathcal{B}(\mathfrak{H})$, leaves invariant the Abelian von Neumann algebra $\mathfrak{N}$ generated by $|T|$. Since (if $\mathfrak{H}_0$ is separable) any Abelian von Neumann algebra $\mathfrak{N}$ can be generated by a single positive operator, it suffices to consider instead the data U and $\mathfrak{N}$. If $\mathfrak{N}$ is maximal, then we may represent it as $L^\infty_{\mathbb{C}}(\mu)$ for some Lebesgue probability space (Ω, Σ, μ), and τ must now define a μ-nonsingular automorphism of (Ω, Σ), and we have the desired representation. Furthermore, if $\mathfrak{N}$ is not maximal, then it enjoys such a representation if and only if it can be extended to a maximal such Abelian algebra on which U still acts as an automorphism.

We shall show that if $\mathfrak{H}$ and $\mathfrak{M}$ are the Hilbert space and von Neumann algebra of our example from Section 3 and $\mathfrak{N}$ is the Abelian subalgebra $\mathrm{Mult}_{L^\infty_{\mathbb{C}}(\mu)}$ then this $\mathfrak{N}$ cannot be extended any further to a larger Abelian von Neumann algebra represented on $\mathfrak{H}$ on which conjugation by $U^{\tau,\Phi}$ still acts as an automorphism.

Indeed, note that any non-trivial Abelian subextension of $\mathfrak{N} \hookrightarrow \mathfrak{M}$ would have to introduce new orthoprojections, and that these must be represented by orthoprojection-valued cocycles over Ω, since they must lie in $\mathfrak{N}' = \mathfrak{M}$. As orthoprojections from some $\alpha^{\tau,\Phi}$-invariant Abelian algebra, they would have to commute with all their conjugates under $U^{\tau,\Phi}$. Thus, the desired non-extendability property for $\mathfrak{N}$ is a consequence of the following.

Theorem 3.1. *For Haar-almost every choice of* (V_1, V_2) *in* $\mathrm{U}(2) \times \mathrm{U}(2)$ *the system cannot support a non-trivial orthoprojection-valued cocycle which commutes with all of its conjugates under* $U^{\tau,\Phi}$.

Proof We will prove this by contradiction; let us suppose that E is such a cocycle. Since the underlying system is ergodic, the rank of E must be μ-almost surely constant, and since E is assumed non-trivial this rank must be 1. Since the subspace of one-dimensional orthoprojections in $\mathrm{Proj}\,\mathfrak{H}_0 = \mathrm{Proj}\,\mathbb{C}^2$ is compact, it follows that for any $\varepsilon > 0$ we may approximate E uniformly by some orthoprojection-valued cocycle taking only finitely many distinct values; and now, since measurable subsets of the Cantor set Ω can be μ-approximated by finite-dimensional sets, we can obtain a further orthoprojection-valued cocycle F that takes only finitely many values, is measurable with respect to some finite algebra of clopen sets $\mathcal{A}$ (say, without loss of generality, the algebra generated by all coordinate-projections of Ω for coordinates in $\{-k, -k+1, \dots, k\}$), and takes values within operator norm ε of those taken by E on all but at most ε of the space Ω in measure.

If F approximates E this well, then so does $\left(\alpha^{\tau,\Phi}\right)^n(F)$ approximate $\left(\alpha^{\tau,\Phi}\right)^n(E)$ for any $n \in \mathbb{Z}$, since conjugation by a unitary preserves the operator norm between two projections and the transformation τ preserves the measure of the error-set. Therefore, if E commutes with all its conjugates, then F must do so approximately on a set of large measure: for any $\eta > 0$, if ε is taken sufficiently small (depending only on the geometry of $\mathrm{U}(2)$), then for any $n \in \mathbb{Z}$ there is some error set $A_n \in \Sigma$ of measure at most 2ε such that for $\omega \in \Omega \setminus A_n$ we have

$$\left\| \left[F(\omega), \left(\alpha^{\tau,\Phi}\right)^n(F)(\omega) \right] \right\| < \delta$$

(note, however, that the error set A_n can vary with n).

However, if we take $n \gg 2k$ then we know that the unitary operators $W_{n,2}$ appearing in Lemma 2.1 corresponding to different values of the intermediate coordinates $\omega_{k+1}, \omega_{k+2}, \dots, \omega_{n-k-1}$ are approximately uniformly distributed in the unitary group $\mathrm{U}(2)$, and so for $\delta > 0$ sufficiently small with probability approaching some fixed positive value these unitary operators do *not* conjugate the second of the above projections to one that δ-approximately commutes with the first. Since this will occur (uniformly in the growth of n) within every cylinder set defined by the coordinates ω_{-k}, ω_{-k+1}, $\ldots$, ω_k and ω_{n-k}, ω_{n+1-k}, $\ldots$, ω_{n+k}, we see that F cannot commute on most of Ω with its far-future conjugates under $\alpha^{\tau,\Phi}$. This is the contradiction we were seeking, and so shows that $\mathfrak{N}$ cannot be extended as described. □

Remark An immediate corollary of the above is that the pair $(U^{\tau,\Phi}, E)$, where E is the spectral measure associated to the Abelian von Neumann algebra $\mathrm{Mult}_{L^\infty_{\mathbb{C}}(\mu)}$, is an irreducible system of imprimitivity over $\mathbb{Z}$ with multiplicity 2, in the sense of Mackey [1]. The construction of such systems for the action of various countable subgroups of $\mathbb{T}$ is treated in Chapter 13 of Nadkarni [2], but we are not aware of such examples of probability-preserving $\mathbb{Z}$-systems to date. In fact, our example gives a little more than irreducibility for our system of imprimitivity of multiplicity greater than 1: not only does no orthoprojection P commute with every $E(\Omega_1)$ and every $U^{\tau,\Phi}$, but in fact any nontrivial P that commutes with every $E(\Omega_1)$ cannot also commute with all of its own conjugates by $E^{\tau,\Phi}$. This tells us that there is no nontrivial Boolean algebra of orthoprojections of $\mathfrak{H}$ that commutes with E and is preserved by $\alpha^{\tau,\Phi}$. ◁

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