

First Name: _____

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Signature: _____

There are a total of 5 problems. Write coherent mathematical statements and show your work on all problems. Please write clearly.

Rules.

1. Start when told to; stop when told to.
2. No calculators, computers, cell phones, notes, books, crib-sheets,...
3. Turn off your cell-phone, pager,...

1	2	3	4	5

TOTAL:

1. (10 points) Let $f : S^1 \rightarrow \mathbb{C}$ be defined by

$$f(x) = \begin{cases} 1, & \text{if } x \in [0, \pi] \\ 0, & \text{if } x \notin [0, \pi] \end{cases}$$

Calculate $\hat{f}(n)$ for all $n \in \mathbb{Z}$.

Hint: make use of the fact that $e^{i\pi n} = (-1)^n$.

ANSWER:

As usual we identify S^1 with the interval $[-\pi, \pi]$.

Now by definition

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx,$$

and for this particular f this is just

$$\frac{1}{2\pi} \int_0^{\pi} 1 e^{-inx} dx.$$

If $n = 0$ this is simply $\frac{1}{2\pi} \int_0^{\pi} 1 dx = \frac{\pi}{2\pi} = \frac{1}{2}$.

If $n \neq 0$ then we have instead

$$\begin{aligned} \frac{1}{2\pi} \int_0^{\pi} e^{-inx} dx &= \frac{1}{2\pi} \left[\frac{-1}{in} e^{-inx} \right]_0^{\pi} \\ &= \frac{-1}{2i\pi n} (e^{-i\pi n} - 1) \\ &= \frac{-1}{2i\pi n} ((-1)^{-n} - 1) \\ &= \begin{cases} \frac{1}{i\pi n} & n \text{ odd} \\ 0 & n \text{ even} \neq 0. \end{cases} \end{aligned}$$

2. (10 points) Suppose that you have a sequence of 2π -periodic functions,

$$K_n : S^1 \rightarrow \mathbb{C}$$

and a constant $M > 0$, such that the following three properties are satisfied:

$$[i] \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(x) dx = 1$$

$$[ii] \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} |K_n(x)| dx \leq M$$

$$[iii] \quad \text{For any } \delta > 0, \lim_{n \rightarrow \infty} \int_{\delta < |x| \leq \pi} |K_n(x)| dx = 0.$$

Show that if $f : S^1 \rightarrow \mathbb{C}$ is a continuous function, then for any $x \in S^1$,

$$f(x) = \lim_{n \rightarrow \infty} \frac{1}{2\pi} \int_{|y| \leq \pi} K_n(y) f(x - y) dy.$$

Note: this convergence is uniform, but you do not need to show this fact.

ANSWER:

We are asked to show that for any $x \in S^1$ and any $\varepsilon > 0$ there is some $n_0 \geq 1$ such that

$$\left| f(x) - \frac{1}{2\pi} \int_{|y| \leq \pi} K_n(y) f(x - y) dy \right| < \varepsilon$$

for all $n \geq n_0$.

First note that since $\frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(y) dy = 1$ (from [i]) we may write $f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(y) f(x) dy$

(moving the constant value $f(x)$ inside the integral), and so the inequality we want to prove becomes

$$\left| \frac{1}{2\pi} \int_{|y| \leq \pi} K_n(y) (f(x) - f(x - y)) dy \right| < \varepsilon.$$

We now see that it will suffice to prove an inequality with the absolute value inside the integral:

$$\frac{1}{2\pi} \int_{|y| \leq \pi} |K_n(y)(f(x) - f(x - y))| dy < \varepsilon.$$

Next, since f is continuous and M is a fixed constant there is some $\delta > 0$ such that $|f(x) - f(x - y)| < \varepsilon/(2M)$ whenever $|y| \leq \delta$. Finally, since f is continuous it is bounded, say by $B > 0$, and now having fixed this δ and B , by [iii] there is some n_0 such that

$$\int_{\delta < |x| \leq \pi} |K_n(x)| dx < \frac{2\pi\varepsilon}{4B}$$

for all $n \geq n_0$.

Putting the above inequalities together we deduce that when $n \geq n_0$ we have

$$\begin{aligned} \frac{1}{2\pi} \int_{|y| \leq \pi} |K_n(y)(f(x) - f(x - y))| dy &= \frac{1}{2\pi} \int_{|y| \leq \delta} |K_n(y)(f(x) - f(x - y))| dy \\ &\quad + \frac{1}{2\pi} \int_{\delta < |y| \leq \pi} |K_n(y)(f(x) - f(x - y))| dy \\ &\leq \frac{1}{2\pi} \int_{|y| \leq \delta} |K_n(y)| |f(x) - f(x - y)| dy \\ &\quad + \frac{1}{2\pi} \int_{\delta < |y| \leq \pi} |K_n(y)| (|f(x)| + |f(x - y)|) dy \\ &\leq \frac{\varepsilon}{2M} \frac{1}{2\pi} \int_{|y| \leq \delta} |K_n(y)| dy + \frac{2B}{2\pi} \int_{\delta < |y| \leq \pi} |K_n(y)| dy \\ &\stackrel{\text{(by [ii])}}{<} \frac{\varepsilon}{2M} M + \frac{2B}{2\pi} \cdot \frac{2\pi\varepsilon}{4B} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

as required.

3. (10 points) If h and g are Riemann integrable, 2π -periodic functions into \mathbb{C} (i.e. $h, g : S^1 \rightarrow \mathbb{C}$), then define

$$(h, g) = \frac{1}{2\pi} \int_{-\pi}^{\pi} h(x) \overline{g(x)} dx,$$

and

$$\|h\|^2 = (h, h) = \frac{1}{2\pi} \int_{-\pi}^{\pi} |h(x)|^2 dx.$$

Define

$$S_N(h)(x) = \sum_{n=-N}^N \hat{h}(n) e^{inx}.$$

Assume that f is Riemann integrable.

- (a) Show that for all $N > 0$

$$(f - S_N(f), S_N(f)) = 0.$$

- (b) Use (a) to show that

$$\|f - S_N(f)\|^2 + \|S_N(f)\|^2 = \|f\|^2.$$

- (c) Use (b) to show

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 \leq \|f\|^2.$$

(Hint: you may use, without proving, that $\|S_N(f)\|^2 = \sum_{n=-N}^N |\hat{f}(n)|^2$.)

ANSWER:

(a) Since $S_N(f)(x) = \sum_{n=-N}^N \hat{f}(n)e^{inx}$ we have

$$\begin{aligned}
(f - S_N(f), S_N(f)) &\stackrel{\text{by defn.}}{=} \frac{1}{2\pi} \int_{-\pi}^{\pi} (f(x) - S_N(f)(x)) \overline{S_N(f)(x)} dx \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \overline{S_N(f)(x)} dx - \frac{1}{2\pi} \int_{-\pi}^{\pi} S_N(f)(x) \overline{S_N(f)(x)} dx \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \sum_{n=-N}^N \overline{\hat{f}(n)e^{inx}} dx - \|S_N(f)\|^2 \\
&= \sum_{n=-N}^N \overline{\hat{f}(n)} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx - \|S_N(f)\|^2 \\
&\stackrel{\text{by defn. of } \hat{f}}{=} \sum_{n=-N}^N \overline{\hat{f}(n)} \hat{f}(n) - \|S_N(f)\|^2 \\
&= \sum_{n=-N}^N |\hat{f}(n)|^2 - \sum_{n=-N}^N |\hat{f}(n)|^2 = 0
\end{aligned}$$

(where we have used that $\|S_N(f)\|^2 = \sum_{n=-N}^N |\hat{f}(n)|^2$ without proof).

(b) Since $(f - S_N(f), S_N(f)) = 0$ by part (a), and hence also $(S_N(f), f - S_N(f)) = \overline{(f - S_N(f), S_N(f))} = 0$, we have

$$\begin{aligned}
\|f\|^2 &= \|f - S_N(f) + S_N(f)\|^2 \\
&= (f - S_N(f) + S_N(f), f - S_N(f) + S_N(f)) \\
&= \|f - S_N(f)\|^2 + (f - S_N(f), S_N(f)) + (S_N(f), f - S_N(f)) + \|f\|^2 \\
&= \|f - S_N(f)\|^2 + \|f\|^2,
\end{aligned}$$

as required.

(c) Since $\|h\|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |h|^2 \geq 0$ for any function h such that $|h|^2$ is integrable,

part (b) and the given equality $\|S_N(f)\|^2 = \sum_{n=-N}^N |\hat{f}(n)|^2$ imply that

$$\|f\|^2 \geq \|S_N(f)\|^2 = \sum_{n=-N}^N |\hat{f}(n)|^2.$$

As $N \rightarrow \infty$ the partial sums on the right-hand side above are non-decreasing, and this inequality shows that they are uniformly bounded above, so they must converge to a limit and it satisfies

$$\|f\|^2 \geq \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2,$$

as required.

4. (10 points) Suppose $f : S^1 \rightarrow \mathbb{C}$ has two derivatives, and there is a constant $M > 0$ such that for any $x \in [-\pi, \pi]$ we have $|f''(x)| < M$. Show that there is a constant C such that for any $n \neq 0$

$$|\hat{f}(n)| \leq \frac{C}{n^2}.$$

ANSWER:

Identifying S^1 with $[-\pi, \pi]$ as usual, by definition we have

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx.$$

Since f is differentiable we may integrate by parts, differentiating $f(x)$ and integrating e^{-inx} , to obtain

$$\begin{aligned} \hat{f}(n) &= \frac{1}{2\pi} \left[\frac{-1}{in} f(x) e^{-inx} \right]_{-\pi}^{\pi} - \frac{-1}{2\pi in} \int_{-\pi}^{\pi} f'(x) e^{-inx} dx \\ &= \frac{-1}{2\pi in} (f(\pi) e^{-in\pi} - f(-\pi) e^{in\pi}) + \frac{1}{2\pi in} \int_{-\pi}^{\pi} f'(x) e^{-inx} dx \\ &= \frac{1}{2\pi in} \int_{-\pi}^{\pi} f'(x) e^{-inx} dx, \end{aligned}$$

where the last equality follows because f is a continuous function on S^1 and so it is 2π -periodic, so that $f(\pi) = f(-\pi)$, and we know that $e^{-in\pi} = (-1)^n = (-1)^{-n} = e^{in\pi}$ for all n .

We now apply exactly the same reasoning to the function f' , which is also assumed differentiable and also satisfies $f'(\pi) = f'(-\pi)$ by 2π -periodicity, to obtain

$$\begin{aligned} \hat{f}(n) &= \frac{1}{2\pi in} \int_{-\pi}^{\pi} f'(x) e^{-inx} dx \\ &= \frac{1}{2\pi (in)^2} \int_{-\pi}^{\pi} f''(x) e^{-inx} dx. \end{aligned}$$

We can now use this and the given bound $|f''(x)| \leq M$ for all x to deduce that

$$\begin{aligned} |\hat{f}(n)| &= \left| \frac{1}{2\pi(in)^2} \int_{-\pi}^{\pi} f''(x)e^{-inx} dx \right| \\ &\leq \frac{1}{2\pi n^2} \int_{-\pi}^{\pi} |f''(x)||e^{-inx}| dx \\ &\leq \frac{M}{2\pi n^2} \int_{-\pi}^{\pi} |e^{-inx}| dx = \frac{M}{n^2}, \end{aligned}$$

which is as required with C equal to the fixed constant M .

5. (10 points) A sequences of functions $\{K_n\}$ satisfying properties [i], [ii], and [iii] in question (2) is called a *Good Kernel*. One example of a *Good Kernel* is the Fejer kernel, given by $K_n(x) = \frac{1}{n} \frac{\sin^2(nx/2)}{\sin^2(x/2)}$. Give **two other** explicit examples of *Good Kernels*. Show that these two examples satisfy the properties [i], [ii], and [iii] of question (2).

ANSWER:

[Note that there are *many, many* possible answers to this question!]

EXAMPLE 1: Let

$$H_n(x) = \begin{cases} 2\pi n & \text{for } |x| \leq 1/(2n) \\ 0 & \text{else} \end{cases}$$

Check this works:

[i]

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} H_n(x) dx = \frac{1}{2\pi} \int_{-1/(2n)}^{1/(2n)} (2\pi n) dx = \frac{2\pi n}{2\pi} (1/(2n) - (-1/(2n))) = 1$$

[ii] Since $H_n(x) \geq 0$ for all x we have

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |H_n(x)| dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_n(x) dx = 1$$

for all n , so [ii] holds with the bound $M = 1$;

[iii] For any given $\delta > 0$, if $n \geq \lceil 1/2\delta \rceil$ then by definition $H_n(x) = 0$ whenever $|x| > \delta \geq 1/(2n)$, so

$$\frac{1}{2\pi} \int_{\delta < |x| \leq \pi} |H_n(x)| dx = 0$$

for all sufficiently large n , so the limit as $n \rightarrow \infty$ certainly equals 0.

EXAMPLE 2: Let

$$L_n(x) = \begin{cases} 2\pi(n - n^2|x|) & \text{for } |x| \leq 1/n \\ 0 & \text{else} \end{cases}$$

Check this works:

[i]

$$\begin{aligned}\frac{1}{2\pi} \int_{-\pi}^{\pi} L_n(x) dx &= \frac{1}{2\pi} \int_{-1/n}^0 2\pi(n + n^2x) dx + \frac{1}{2\pi} \int_0^{1/n} 2\pi(n - n^2x) dx \\ &= \left(\frac{n}{n} + n^2 \left[\frac{x^2}{2} \right]_{-1/n}^0 + \frac{n}{n} - n^2 \left[\frac{x^2}{2} \right]_0^{1/n} \right) \\ &= \left(\frac{n}{n} - n^2(1/n)^2/2 + \frac{n}{n} - n^2(1/n)^2/2 \right) \\ &= 1,\end{aligned}$$

as required.

[ii] exactly as for Example 1, since $L_n(x) \geq 0$ for all x ;

[iii] exactly as for Example 1, since for any given $\delta > 0$, if $n \geq \lceil 1/\delta \rceil$ then by definition $L_n(x) = 0$ whenever $|x| > \delta \geq 1/n$.