245B, Winter 2009, Assignment 7:
Notes and selected model answers

(Model solutions follow question numbers in **bold**.)

Folland Chapter 4

3. Suppose that X is a metric space. Then it is clear that individual points are closed, so X is T_1 [technically this is part of the definition of normality and as such it needs to be checked, although it's not the heart of this question and wasn't worth any marks on the homework].

Now suppose that $A, B \subseteq X$ are closed and disjoint. We need to find disjoint open sets $U \supseteq A$ and $V \supseteq B$. Let us take $U := \{x \in X : \ \rho(x, A) < \rho(x, B)\}$ and $V := \{x :\in X : \ \rho(x, A) > \rho(x, B)\}$ and show that these will do.

Firstly, they are manifestly disjoint, since any $x \in U \cap V$ would have to satisfy two contradictory inequalities.

Secondly, any $x \in A$ has $\rho(x, A) = 0$, but since $x \notin B$ (because $A \cap B = \emptyset$) and B is *closed*, x actually lies in the interior of $X \setminus B$ and therefore lies at the centre of some open ball disjoint from B, and so in particular we have $\rho(x, B) > 0$. Therefore $x \in U$, and so since $x \in A$ was arbitrary we have $A \subseteq U$. An exactly analogous argument gives $B \subseteq V$.

Finally, suppose that $x \in U$, so $r := \rho(x, B) - \rho(x, A) > 0$. Then if $y \in B(x, r/3)$ the triangle inequality gives

- for any $b \in B$, $\rho(y, b) \geq \rho(x, b) - \rho(x, y) > \rho(x, B) - r/3$, so taking the infimum over $b \in B$ gives $\rho(y, B) \geq \rho(x, B) - r/3$, and
- for any $a \in A$, $\rho(y, a) \leq \rho(x, a) + \rho(y, x) < \rho(x, a) + r/3$, so taking the infimum over $a \in A$ gives $\rho(y, A) \leq \rho(x, A) + r/3 < \rho(x, B) - r/3 \leq \rho(y, B)$,

and so $y \in U$ also [note that slightly more careful estimates allow one to take the radius $r/2$ above, but of course this isn't important]. Therefore $B(x, r/3)$ is a neighbourhood of x lying entirely within U, so U is open, and exactly analogously V is open.

46. [There are many correct ways to present this proof, mostly quite similar one to another, and what follows is just a sample.]

Theorem (Locally Compact Tietze Extension Theorem)**.** *Suppose that X is an LCH space and $K \subset X$ is compact. If $f \in C(K)$, there exists $F \in C(X)$ such that $F|_K = f$. Moreover, F may be taken to vanish outside a compact set.*

Proof By Folland's Proposition 4.31 [this appeal is permitted, since it mimics the first step in the proof of Folland's Lemma 4.32, as instructed], we can find an open set $V \supseteq K$ such that $\bar{V}$ is compact.

We will define F in two steps, first extending f to $\bar{V}$ and then to the whole of X, where the latter extension is simply by $F|_{X\setminus\bar{V}} = 0$ (so guaranteeing that F vanishes outside the compact set $\bar{V}$).

Now, in order that this be continuous at the boundary $\partial V := \bar{V} \setminus V$, we must first obtain a continuous extension to $\bar{V}$ that vanishes on ∂V. To this end, first define $g : K \cup \partial V \to \mathbb{C}$ by $g|_K = f$ and $g|_{\partial V} = 0$; this is unambiguous because $K \cap \partial V = \emptyset$.

In fact, this g is continuous on $K \cup \partial V$, because for any $x \in K \cup \partial V$ and $\varepsilon > 0$,

- either $x \in K$, in which case $x \in V$, an open set, and so since f is continuous we can choose a neighbourhood $U \ni x$ such that $f(K \cap U) \supseteq B(f(x), \varepsilon)$, and now $(K \cap U) = (K \cup \partial V) \cap (U \cap V)$ is a neighbourhood of x in $K \cup \partial V$ such that $g((K \cup \partial V) \cap (U \cap V)) \subseteq B(g(x), \varepsilon)$,
- or $x \in \partial V$, in which case (because x lies outside the closed set K) there is some neighbourhood U of x disjoint from K, and now $(K \cup \partial V) \cap U = \partial V \cap U$ and so $g((K \cup \partial V) \cap U) = g(\partial V \cap U) = \{0\} \subseteq B(0, \varepsilon) = B(g(x), \varepsilon)$,

so any $x \in K \cup \partial V$ admits a relative neighbourhood in $K \cup \partial V$ in which g takes values within ε of $g(x)$.

Therefore, applying the Tietze Extension Theorem to the compact (hence normal) space $\bar{V}$ there is a continuous function $h : \bar{V} \to \mathbb{C}$ with $h|_{K\cup\partial V} = g$, and now we define F by $F|_{\bar{V}} = h$ and $F|_{X\setminus\bar{V}} = 0$. It is clear that $F|_K = h|_K = f$, and F is continuous because for any point x either $x \in V$ (in which case F agrees with the continuous function h on some neighbourhood of x), or $x \in \partial V$ (in which case for any $\varepsilon > 0$ we can find a neighbourhood $U \ni x$ such that $h(\bar{V} \cap U) \subseteq B(0, \varepsilon)$ and also $F((X \setminus \bar{V}) \cap U) = \{0\} \subseteq B(0, \varepsilon)$, so in fact $F(U) \subseteq B(0, \varepsilon)$), or $x \in X \setminus \bar{V}$ in which case it has some neighbourhood on which F vanishes completely. Therefore this F has all the desired properties.

68. [This should follow rather mechanically from the Stone-Weierstrass Theorem, although with some care required to verify carefully all of the requirements of that theorem.]

Folland Chapter 5

53. a. Since $T_n \to T$ strongly, it follows from Exercise 47a. that $M := \sup_n \|T_n\| < \infty$. Now if also $\|x_n - x\| \to 0$ the triangle inequality gives

$$\|T_n x_n - Tx\| \leq \|T_n(x_n - x)\| + \|T_n x - Tx\| \leq M\|x_n - x\| + \|T_n x - Tx\|,$$

and on the right-hand side above the first term tends to 0 as $n \to \infty$ because $\|x_n - x\| \to 0$ and M is fixed, and the second term also tends to 0 because by assumption $T_n \to T$ strongly.

b. Given also that $S_n \to S$ strongly, for any $x \in X$ we have $\|S_n x - Sx\| \to 0$ (by the definition of strong convergence) and so, applying part a. to the sequence of vectors $S_n x \to Sx$, we now have $\|T_n(S_n x) - T(Sx)\| \to 0$, so $T_n S_n x \to TSx$ in norm. Since x was arbitrary this tells us that $T_n S_n \to TS$ strongly, as required.

Main class URL: `http://www.math.ucla.edu/~tao/245b.1.09w/`

TA class URL:

`http://www.math.ucla.edu/~timaustin/teaching_245B_W09`

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