245B, Winter 2009, Assignment 6:
Notes and selected model answers

(Model solutions follow question numbers in **bold**.)

Folland Chapter 4

63. We first recall that since $[0,1]$ is compact, any $f \in C[0,1]$ is bounded, and that since $[0,1]\times[0,1]$ is compact, the continuous function $K : [0,1]\times[0,1] \to \mathbb{C}$ is bounded and is actually *uniformly* continuous [this is crucial for this question].

From the second of these observations it follows that for any $\varepsilon > 0$ we can choose $\delta > 0$ so small that $|K(x,y)-K(x',y')| < \varepsilon$ whenever $\max\{|x-x'|,|y-y'|\} < \delta$, and so certainly when $y' = y$ and $|x-x'| < \delta$. Therefore given any f, using that $\|f\|_{\mathrm{u}}$ is finite, if $|x-x'| < \delta$ then we also have

$$\begin{aligned}|Tf(x) - Tf(x')| &= \Big|\int_0^1 K(x,y)f(y)\,\mathrm{d}y - \int_0^1 K(x',y)f(y)\,\mathrm{d}y\Big| \\ &\leq \int_0^1 |K(x,y)-K(x',y)||f(y)|\,\mathrm{d}y < \varepsilon\int_0^1 |f(y)|\,\mathrm{d}y \leq \varepsilon\|f\|_\infty.\end{aligned}$$

Since ε was chosen arbitrarily this shows that for any fixed f the output function Tf is continuous.

Moreover, in case $\|f\|_{\mathrm{u}} \leq 1$ the above inequality specializes to the assertion that $|Tf(x) - Tf(x')| < \varepsilon$ whenever $|x-x'| < \delta$ without any further restrictions on f. Since this δ was initially chosen without reference to f, it follows that the collection $\{Tf : \|f\|_{\mathrm{u}} \leq 1\}$ is actually *equi*continuous. In addition we always have

$$|Tf(x)| \leq \Big|\int_0^1 K(x,y)f(y)\,\mathrm{d}y\Big| \leq \|K\|_{\mathrm{u}}\|f\|_{\mathrm{u}} \leq \|K\|_{\mathrm{u}}$$

for any f with $\|f\|_{\mathrm{u}} \leq 1$, so this collection is also pointwise bounded, and so the Arzelá-Ascoli Theorem tells us that it is precompact in $C[0,1]$.

Folland Chapter 5

47. **a.** We are given that $f(T_n x) \to f(Tx)$ for any $x \in X$ and $f \in X^*$, and need to deduce that the sequence of norms $\|T_n\|$ is bounded. This should follow from the Uniform Boundedness Principle, but a priori we are not given the boundedness of any sequence of norms even of individual vectors. We overcome this difficulty by applying that principle *twice*.

First, we recall (from Folland's Theorem 5.8d., for example, which is itself a corollary of the Hahn-Banach Theorem) that the map $X \to X^{**} : x \mapsto \widehat{x}$ defined by $\widehat{x}(f) := f(x)$ is an isometric embedding, and so we can re-write the given condition as asserting that $\widehat{T_n x}(f) \to \widehat{Tx}(f)$ for any $x \in X$ and $f \in X^*$. This implies, in particular, that for any fixed x, for any fixed f the sequence $|\widehat{T_n x}(f)|$ is bounded (since it converges to $|\widehat{Tx}(f)|$), and so applying the Uniform Boundedness Principle to the operators $\widehat{T_n x} \in \mathcal{L}(X^*, \mathbb{C})$ and using the above-mentioned isometricity of the embedding $X \hookrightarrow X^{**}$ gives that $\sup_n \|\widehat{T_n x}\| = \sup_n \|T_n x\| < \infty$.

Now since this holds for every $x \in X$ a second appeal to the Uniform Boundedness Principle gives the desired bound $\sup_n \|T_n\| < \infty$.

Finally note that strong convergence for operators implies weak convergence, so the above argument actually completes the whole of part a.

b. Suppose first that x_n is a sequence in X that converges weakly to $x \in X$. Then for any $f \in X^*$ we have $|\widehat{x_n}(f)| = |f(x_n)| \to |f(x)|$ as $n \to \infty$, and so the first sequence of values is bounded. Applying the Uniform Boundedness Principle to the sequence $(\widehat{x_n})_n$ in X^{**} gives $\sup_n \|\widehat{x_n}\| = \sup_n \|x_n\| < \infty$.

Even more simply, if f_n converges weak* to f in X^*, then $|f_n(x)| \to |f(x)|$ for every $x \in X$, so the sequence $|f_n(x)|$ is bounded for every x and so the Uniform Boundedness Principle gives $\sup_n \|f_n\| < \infty$, as required.

48. [This requires much the same ideas as 47 and 51: in general, weak neighbourhoods witnessing that a given set is weakly open/closed are constructed by some judicious appeal to the Hahn-Banach Theorem.]

51. Since the norm topology of X is at least as strong as the weak topology, any open subset for the latter is also open for the former; and taking complements it follows that any closed subset for the latter is also a closed subset for the former.

It remains to show that for a vector sub*space* $M \leq X$ the reverse is also true: if M is norm-closed then it is weakly closed. To this end suppose that M is norm-closed and that $x \in X \setminus M$ (if $M = X$ then the fact that M is weakly closed is immediate). Then Folland's Theorem 5.8a. (a corollary of the Hahn-Banach Theorem) gives some $f \in X^*$ with $f|_M = 0$ but $f(x) \neq 0$. Since the weak topology is precisely the weakest topology with respect to which all members of X^* are continuous, it follows that $f^{-1}(\mathbb{C} \setminus \{0\})$ is a weakly open subset of X (as the continuous inverse image of an open subset of $\mathbb{C}$) that contains x but is disjoint from M (because $f|_M = 0$), so this x actually lies in the weak interior of $X \setminus M$. Since x was arbitrary, M is weakly closed.

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