245B, Winter 2009, Assignment 5:
Notes and selected model answers

(Model solutions follow question numbers in **bold**.)

Folland Chapter 4

28. **a.** We must verify that $\mathcal{T}$ satisfies the axioms of a topology.

1. We always have $\pi^{-1}(\emptyset) = \emptyset$ and $\pi^{-1}(\tilde{X}) = X$, and both of these pre-images are open in X, so $\emptyset, \tilde{X} \in \mathcal{T}$;
2. If $\mathcal{U}$ is any collection of members of $\mathcal{T}$, then by definition $\{\pi^{-1}(U) : U \in \mathcal{U}\}$ is a collection of open subsets of X. Therefore $\bigcup_{U\in\mathcal{U}} \pi^{-1}(U)$ is a union of open sets in X and so is itself open, but since
$$\pi^{-1}\Big(\bigcup \mathcal{U}\Big) = \bigcup_{U\in\mathcal{U}} \pi^{-1}(U)$$
this tells us that $\pi^{-1}\big(\bigcup \mathcal{U}\big)$ is open, and so $\bigcup \mathcal{U} \in \mathcal{T}$;
3. Exactly similarly, if $\mathcal{U}$ is a finite family of members of $\mathcal{T}$ then
$$\pi^{-1}\Big(\bigcap \mathcal{U}\Big) = \bigcap_{U\in\mathcal{U}} \pi^{-1}(U)$$
is now a *finite* intersection of open subsets of X, so is itself open, and so the finite intersection $\bigcap \mathcal{U}$ also lies in $\mathcal{T}$.

b. Suppose that Y is another topological space and $f : \tilde{X} \to Y$ a function.

First note that if $U \subseteq \tilde{X}$ is open (i.e., it lies in the family $\mathcal{T}$ that we have shown above to be a topology) then by the definition of $\mathcal{T}$ we have that $\pi^{-1}(U)$ is open in X. Phrased differently, this tells us that π is continuous, and so if f is also continuous then the composition $f \circ \pi$ is continuous.

On the other hand, if $f \circ \pi$ is continuous and $V \subseteq Y$ is open then $(f \circ \pi)^{-1}(V) = \pi^{-1}(f^{-1}(V))$ must be open in X, and so by definition $f^{-1}(V)$ lies in $\mathcal{T}$: this shows that any f-preimage of an open set in Y is open in $\tilde{X}$, so f is also continuous.

43. Suppose that $(a_{n_k})_{k\geq 1}$ is any subsequence of $(a_n)_n$; we will show that it cannot be pointwise convergent.

Indeed, simply setting

$$x := \sum_{k\geq 1} 2^{-n_{2k}},$$

this sequence clearly converges to a positive real, and (since the series must be dominated by $\sum_j 2^{-j}$) this real lies in $[0,1]$. Moreover it must be a dyadic irrational, since the binary expansion given for it above contains both infinitely many 1s (at positions n_{2k}, $k \geq 1$) and infinitely many 0s (at positions n_{2k-1}, $k \geq 1$, and any other positions not in the sequence $\{n_1, n_2, \ldots\}$). Therefore its binary digits are unambiguously given by the above expansion, and so we have

$$a_{n_j}(x) = \begin{cases} 1 & \text{if } j \text{ even} \\ 0 & \text{if } j \text{ odd,} \end{cases}$$

which sequence does not converge as $j \to \infty$. Thus this point x witnesses that our subsequence does not converge pointwise.

59. Suppose that X_1, X_2, ..., X_n are locally compact spaces and that $x = (x_1, x_2, \ldots, x_n) \in X_1 \times X_2 \times \cdots \times X_n$. We must show that x has a compact neighbourhood in the product topology.

Since each X_i is locally compact we can choose for each x_i, $i = 1, 2, \ldots, n$, a compact neighbourhood K_i (that is, K_i is compact and x_i lies in its interior), and now $K_1^\circ \times K_2^\circ \times \cdots \times K_n^\circ$ is an open subset of $X_1 \times X_2 \times \cdots \times X_n$ containing x [it is acceptable simply to state that a product of finitely many open sets is open, but to be safe one can write that $K_1^\circ \times K_2^\circ \times \cdots \times K_n^\circ = \pi_1^{-1}(K_1^\circ) \cap \pi_2^{-1}(K_2^\circ) \cap \cdots \cap \pi_n^{-1}(K_n^\circ)$].

Since also $K_1^\circ \times K_2^\circ \times \cdots \times K_n^\circ \subseteq K_1 \times K_2 \times \cdots \times K_n$, it will suffice to show that this latter set is compact, for then that will also be true of $\overline{K_1^\circ \times K_2^\circ \times \cdots \times K_n^\circ}$. Now, Tychonoff's Theorem tells us that it is a compact space for the product of the relative topologies on each K_i. To complete the proof, let $\mathcal{U}$ be an open cover of this set in $X_1 \times X_2 \times \cdots \times X_n$. Then the collection $\{U \cap (K_1 \times K_2 \times \cdots \times K_n) : U \in \mathcal{U}\}$ is a cover of $K_1 \times K_2 \times \cdots \times K_n$ by subsets of this smaller space, and it is clear that these are still open for the product of the subspace topologies [this can just be stated here]. Hence this family of intersections has a finite subcover, and choosing a corresponding finite subfamily of $\mathcal{U}$ completes the proof that $K_1 \times K_2 \times \cdots \times K_n$ is compact.

65. [This is most quickly done by an appeal to the Arzelá-Ascoli Theorem II. Note that for this question it's best to write out at least a sketch proof that the resulting limit function is not only continuous but holomorphic, and that one needs to be careful to keep all the action inside the domain U: for example, taking only contours that lie strictly inside U for contour integrals.]

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TA class URL:

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Tim Austin
MS 3931
timaustin@math.ucla.edu