245B, Winter 2009, Assignment 4:
Notes and selected model answers

(Model solutions follow question numbers in **bold**.)

Folland Chapter 5

22. **c.** We assume from parts a. and b. that we have already defined the adjoint operator $T^\dagger \in L(Y^*, X^*)$ associated to a bounded linear operator $T \in L(X, Y)$. We will prove that $T^\dagger$ is injective if and only if T has dense range.

($\Rightarrow$) We prove the contrapositive. Suppose that $\overline{R(T)} \subsetneqq Y$. Then since $\overline{R(T)}$ is a proper closed subspace of Y, by the Hahn-Banach Theorem there exists some $f \in Y^*$ with $f \neq 0$ but $f|_{\overline{R(T)}} = 0$, and hence in particular $f|_{R(T)} = 0$. This latter restrictions tells us that for any $x \in X$ we have $T^\dagger f(x) =_{\text{def}} f(Tx) = 0$, and thus that $T^\dagger f = 0$, so $T^\dagger$ is not injective.

($\Leftarrow$) Once again we prove the contrapositive. If $T^\dagger f = T^\dagger g$ for some $f \neq g$ in Y^*, then $T^\dagger h = 0$ for $h := f - g$. This tells us that $h(Tx) = T^\dagger h(x) = 0$ for all $x \in X$, so since $h \neq 0$ we deduce that $R(T)$ is contained in the proper closed subspace $\ker h \subsetneqq Y$, and so $R(T)$ is not dense in Y.

30. **a.** The sequence of functions $f_n : x \mapsto \sqrt{(x - 1/2)^2 + 1/n^2}$ is easily verified to be continuously differentiable (indeed, smooth) on $[0, 1]$, but they converge uniformly to $f : x \mapsto |x - 1/2|$, which is not differentiable at $1/2$.

b. To see that (d/dx) is closed, suppose that $\big((f_n, g_n)\big)_n$ is a sequence in $\mathrm{graph}(d/dx) \subseteq X \times Y$ converging to (f, g). Then $f_n \to f$ uniformly and $g_n = df_n/dx \to g$ uniformly. It follows from the Fundamental Theorem of Calculus and the Dominated Convergence Theorem [or more elementary estimates for uniform convergence of functions] that

$$f_n(x) - f_n(0) = \int_0^x g_n(t)\,\mathrm{d}t \to \int_0^x g(t)\,\mathrm{d}t,$$

and so since the left-hand side tends to $f(x) - f(0)$, we have $f(x) - f(0) = \int_0^x g(t)\,\mathrm{d}t$. Appealing again to the Fundamental Theorem of Calculus now gives that $g = df/dx$ and hence $(f, g) \in \mathrm{graph}(d/dx)$, so the graph is closed.

On the other hand, taking $f_n : x \mapsto x^n$ we observe that $f_n \in C^1([0, 1])$ has $\|f_n\|_{\mathrm{u}} = 1$ but $\|df_n/dx\|_{\mathrm{u}} = n$, so (d/dx) is not bounded.

32. Let X_i denote the normed vector space $(X, \|\cdot\|_i)$ for $i = 1, 2$; by assumption these are both complete and hence Banach spaces. Now let $T : X_2 \to X_1$ be the formal identity (that is, $T(x) := x$): since $\|\cdot\|_1 \leq \|\cdot\|_2$ we have $\|T(x)\|_1 = \|x\|_1 \leq \|x\|_2$ for all x and hence $\|T\|_{\mathrm{op}} \leq 1$. So T is a bounded linear operator between two Banach spaces that is manifestly bijective, and so by Folland's Corollary 5.11 to the Open Mapping Theorem it has a bounded inverse, and hence for any $x = T(x) \in X_1$ we have $\|x\|_2 \leq \|T^{-1}\|_{\mathrm{op}}\|x\|_1$, completing the desired equivalence of norms.

42. [Note that one can either prove that each E_n itself is closed — which it is — or work throughout with $\overline{E_n}$, starting in part a. by proving that this closure has empty interior. In fact the latter approach seems to be slightly easier, but the basic ideas involved are the same.]

Main class URL: `http://www.math.ucla.edu/~tao/245b.1.09w/`

TA class URL:

`http://www.math.ucla.edu/~timaustin/teaching_245B_W09`

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