245B, Winter 2009, Assignment 3:
Notes and selected model answers

(Model solutions follow question numbers in **bold**.)

Folland Chapter 5

19. [Note that for part b. you'll need to show that any open ball containing $0 \in X$ contains a sequence with no convergent subsequence. If the radius of such a ball is $\varepsilon > 0$, you can obtain such a sequence by scaling the sequence found in part a. by $\varepsilon/2$.]

20. Let K be the scalar field of X ($\mathbb{R}$ or $\mathbb{C}$). Since M is finite-dimensional we can choose for it a finite basis, say e_1, e_2, …, e_m. Now define $f_j \in M^*$ by setting $f_j(e_i) := \delta_{i,j}$ for $j \leq m$ and extending by linearity. As linear functionals on a finite-dimensional normed space these f_j are all continuous. Consequently the Hahn-Banach Theorem gives us for each $j \leq m$ an extension $g_j \in X^*$ with $g_j|M = f_j$.

Finally, let $N := \bigcap_{j \leq m} \ker g_j$; we show that this works. On the one hand, if $x \in M \cap N$, then in particular $x = \lambda_1 e_1 + \cdots + \lambda_m e_m$ for some $\lambda_1, \lambda_2, \ldots, \lambda_m \in K$. Now since $g_j|M = f_j$ and we also have $x \in \ker g_j$ for every $j \leq m$, we have $0 = g_j(x) = f_j(x) = \lambda_j$ for each $j \leq m$: that is, $x = 0$.

On the other hand, for any $x \in X$ we can write $x = \sum_{j=1}^m g_j(x)e_j + \big(x - \sum_{j=1}^m g_j(x)e_j\big)$, and now $\sum_{j=1}^m g_j(x)e_j$ is manifestly in M while for any $\ell \leq m$ we have

$$g_\ell\Big(x - \sum_{j=1}^m g_j(x)e_j\Big) = g_\ell(x) - \sum_{j=1}^m g_j(x)g_\ell(e_j) = g_\ell(x) - g_\ell(x) = 0,$$

so the remainder term $x - \sum_{j=1}^m g_j(x)e_j$ lies in N, and thus $X = M + N$.

56. [The main idea here appears in the proof that $(E^\perp)^\perp = \overline{E}$: the point is that if this fails, then we can find some $y \in (E^\perp)^\perp \setminus \overline{E}$, and now by a Gram-Schmidt-like

procedure we can adjust y so that it is actually *orthogonal* to $\overline{E}$ and nonzero; hence $0 \neq y \in E^{\perp} \cap (E^{\perp})^{\perp}$, which leads to a contradiction.]

62. a. Fix $\varepsilon > 0$. By Folland's Theorem 2.10 b. there is a sequence $(\phi_n)_n$ of simple functions on $[0,1]$ such that $0 \leq |\phi_1| \leq |\phi_2| \leq \ldots \leq |f|$ and $\phi_n \to f$ pointwise. Since $|\phi_n - f|^2 \leq (|\phi_n| + |f|)^2 \leq 4|f|^2$ and $|\phi_n - f|^2 \to 0$ pointwise, by the Dominated Convergence Theorem we have $\int |\phi_n - f|^2 \leq \varepsilon^2$ for some sufficiently large n.

Let $\phi_n = \sum_j \alpha_j \chi_{E_j}$. Now Folland's Proposition 1.20 tells us that each χ_{E_j} can in turn be approximated in $\|\cdot\|_1$ by a finite sum $\sum_p \chi_{I_{j,p}}$ of indicator functions of disjoint open intervals. Since both of these functions are bounded by 1, this approximation is also small in $\|\cdot\|_2$, and so choosing a sufficiently close such approximant for each χ_{E_j} and summing gives an $\|\cdot\|_2$-approximation to f within 2ε by a finite linear combination $\sum_q \beta_q \chi_{I_q}$ of indicator functions of open intervals.

Finally, for any nonempty open interval (a,b) we may approximate its indicator function arbitrarily well in $\|\cdot\|_2$ by a continuous function: for example, for $\eta < (b-a)/2$ the function

$$g : x \mapsto \begin{cases} 0 & 0 \leq x \leq a \\ (x-a)/\eta & a < x < a+\eta \\ 1 & a+\eta \leq x \leq b-\eta \\ (b-x)/\eta & b-\eta < x < b \\ 0 & b \leq x \leq 1 \end{cases}$$

is continuous and $\|\cdot\|_2$-approximates $\chi_{(a,b)}$ within $\sqrt{2\eta}$. Choosing a continuous function g_q that $\|\cdot\|_2$-approximates χ_{I_q} within $\varepsilon / \big(\sum_q |\beta_q| \big)$ and adding these now gives an $\|\cdot\|_2$-approximation to f within 3ε. Since $\varepsilon > 0$ was arbitrary this completes the proof.

[An alternative approach is first to approximate f by some truncated function $f\chi_{\{|f| \leq M\}}$ in $\|\cdot\|_2$ using the monotone convergence theorem, and then use Folland's Theorem 2.26 as a black box to approximate $f\chi_{\{|f| \leq M\}}$ — which now lies in L^1 — in $\|\cdot\|_1$ by a simple function made up from indicator functions of open intervals, where the second approximation is chosen so small in terms of M that it then still gives a good $\|\cdot\|_2$ approximation as well.]

b. Given $f \in L^2([0,1])$ and $\varepsilon > 0$, by part a. there is some $g \in C([0,1])$ with $\|f - g\|_2 \leq \varepsilon$, and now by the Weierstrass Polynomial Approximation Theorem there is some polynomial p such that $\|g - p\|_{\mathrm{u}} \leq \varepsilon$, and so since $\|\cdot\|_2 \leq \|\cdot\|_{\mathrm{u}}$ for functions on $[0,1]$ this p is also an $\|\cdot\|_2$-approximation to f within 2ε. Since $\varepsilon > 0$ was arbitrary this completes the proof.

c. Given $f \in L^2([0,1])$ and $\varepsilon > 0$, by part b. there is some polynomial $p(x) = \sum_{j=0}^m a_j x^j$ such that $\|f - p\|_2 \leq \varepsilon$, and now by replacing each coefficient a_j with a sufficiently close rational we can find a polynomial p' with rational coefficients such that $\|f - p'\|_2 \leq 2\varepsilon$. On the other hand the set of polynomials with rational coefficients is countable, so since $\varepsilon > 0$ was arbitrary we have found a countable dense set in L^2.

[Alternatively, re-use the approximations to f by simple functions built from open intervals obtained in part a., showing that each of the constituent indicator functions can be approximated by an indicator function of an interval with rational endpoints and that once again all the coefficients can be approximated by rationals.]

d. It is clear that the approximations by rational polynomials obtained in part c. above holds good when considered for functions on $[0,1)$, rather than $[0,1]$ (since $\{1\}$ is a Lebesgue-negligible set), and now by the translation invariance of Lebesgue measure on $\mathbb{R}$, the same holds for any half-open unit interval $[n, n+1)$. Finally, writing $\mathbb{R} = \bigcup_{n \in \mathbb{Z}} [n, n+1)$, the conclusion of Exercise 5.60 tells us that $L^2(\mathbb{R})$ is separable.

e. We prove that $L^2(\mathbb{R}^n)$ is separable by induction on n. The base case $n = 1$ is the conclusion of part d. above, so suppose now that we have proved the result for $L^2(\mathbb{R}^i)$ for all $i \leq n$ and wish to prove it for $L^2(\mathbb{R}^{n+1})$. By Folland's Proposition 5.29 it suffices to show that this has a countable basis, and also by that proposition we know from their separability that both $L^2(\mathbb{R})$ and $L^2(\mathbb{R}^n)$ do have countable orthonormal bases, say $\{f_n\}_n$ and $\{g_n\}_n$ respectively. Finally, identifying $\mathbb{R}^{n+1}$ with Lebesgue measure as $\mathbb{R}^n \times \mathbb{R}$ with the product of Lebesgue measures, the conclusion of Exercise 5.61 tells us that $\{f_n \otimes g_m\}_{n,m}$ is an orthonormal basis for $L^2(\mathbb{R}^{n+1})$, so since this set is countable the proof is complete.

Folland Chapter 6

6. [This is unfortunately a very fiddly question. The main intuition behind it is that for a function such as $x^{-a}|\log x|^b$, the power of x controls the 'coarse' asymptotic behaviour of the function as $x \to \infty$ or $x \downarrow 0$, and so gives p^{th}-power integrability or non-integrability of the function in the desired range except possibly at the end-points of that range, and now the power of $|\log x|$, whose asymptotic behaviour is slower and more delicate, can be used to fine-tune an example to either include or exclude the end-points of the desired interval of p. In addition, note that there is a lot of flexibility in how one uses these basic ideas: for example, the construction can be made much easier by chopping up $(0, \infty)$ into certain subintervals

and simply picking different examples on those subintervals, and also by making only crude estimates above/below to show the integrability/non-integrability of certain functions, rather than trying to evaluate their integrals exactly.]

12. When $\dim L^p \leq 1$ then L^p is either a single point $\{0\}$, which carries the trivial norm and trivial inner product, or a single line $\{\lambda f : \lambda \in K\}$, in which case the norm is a positive multiple of $|\lambda| = \sqrt{\lambda\overline{\lambda}}$ and this does arise from an inner product.

Suppose, then, that $\dim L^p \geq 2$. We first treat the case $p < \infty$. We will suppose that the parallelogram law holds and prove that $p = 2$.

The lower bound on dimension requires that the underlying measure space $(X, \mathcal{M}, \mu)$ contain at least two sets of finite positive measure. Indeed, if there is any non-zero $f \in L^p(\mu)$ then the set

$$\{f \neq 0\} = \bigcup_{n \geq 1} \{|f| \geq 1/n\}$$

must have positive measure, and hence we must have $\mu\{|f| \geq 1/n\} > 0$ for n sufficiently large, and moreover

$$\infty > \int |f|^p \geq \int_{\{|f| \geq 1/n\}} \frac{1}{n^p} = \frac{1}{n^p}\mu\{|f| \geq 1/n\},$$

so in fact $\infty > \mu\{|f| \geq 1/n\} > 0$ for such n. This gives at least one set of finite positive measure. If there were only one such set, say A, then the above reasoning would shows that for any $f \in L^p$ and $t > 0$ the set $\{|f| \geq t\}$ is either negligible or a set of finite positive measure, and in the latter case it must be equal to A, and from this it follows that $f = t_0 1_A$ μ-a.s., where $t_0 := \inf\{t > 0 : \mu\{|f| \geq t\} = 0\}$. Since this applies to any f, we have found in this case that L^p is only one-dimensional, contradicting our assumption above.

Hence we may assume two sets E, F of finite positive measure and with $\mu(E \triangle F) > 0$, and now by replacing either E with $E \setminus F$ or F with $F \setminus E$ we may assume they are disjoint. Finally, letting $f := \big(\frac{1}{\mu(E)^{1/p}} 1_E\big)$ and $g := \big(\frac{1}{\mu F^{1/p}} 1_F\big)$ and appealing to the parallelogram law, we compute that

$$\begin{aligned}
&\|f+g\|_p^2 + \|f-g\|_p^2 \\
&= \Big(\int_E \Big|\frac{1}{\mu(E)^{1/p}}\Big|^p + \int_F \Big|\frac{1}{\mu(F)^{1/p}}\Big|^p\Big)^{2/p} + \Big(\int_E \Big|\frac{1}{\mu(E)^{1/p}}\Big|^p + \int_F \Big| - \frac{1}{\mu(F)^{1/p}}\Big|^p\Big)^{2/p} \\
&= \Big(\frac{1}{\mu(E)}\mu(E) + \frac{1}{\mu(F)}\mu(F)\Big)^{2/p} = 2^{2/p} \\
&= 2\|f\|_p^2 + 2\|g\|_p^2 = 2\Big(\int_E \Big|\frac{1}{\mu(E)^{1/p}}\Big|^p\Big)^{2/p} + 2\Big(\int_F \Big|\frac{1}{\mu(F)^{1/p}}\Big|^p\Big)^{2/p} \\
&= 2 + 2 = 4,
\end{aligned}$$

and so we must have $p = 2$.

Similarly, when $p = \infty$, we may no longer have any sets of finite positive measure, but we can at least conclude that there are two non-negligible sets E, F such that $E \triangle F$ is also non-negligible (again, since the support of any non-zero bounded measurable function gives us one such set, and if there were only one then all functions in L^∞ would be multiples of its indicator function, contradicting the dimension lower bound). Once again simply by cutting down either E or F we can assume $E \cap F = \emptyset$, and now letting $f := 1_E$ and $g := 1_F$ we conclude in this case that $\|f+g\|_\infty^2 + \|f-g\|_\infty^2 = 1 + 1 = 3$ whereas $2\|f\|_\infty^2 + 2\|g\|_\infty^2 = 2 + 2 = 4$, so the parallelogram law also does not hold when $p = \infty$, as required.

Main class URL: `http://www.math.ucla.edu/~tao/245b.1.09w/`

TA class URL:

`http://www.math.ucla.edu/~timaustin/teaching_245B_W09`

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