245B, Winter 2009, Assignment 2:
Notes and selected model answers

(Model solutions follow question numbers in **bold**.)

Folland Chapter 5

7. a. We are given $T \in \mathcal{L}(\mathcal{X})$ such that $\|I-T\| < 1$. Let $R_n := \sum_{n=0}^{n}(I-T)^n$ for $n \geq 1$. This is a Cauchy sequence in $\mathcal{L}(\mathcal{X})$, since for any $m > n$ we have

$$\|R_n - R_m\| = \Big\| \sum_{i=n+1}^{m} (I-T)^i \Big\| \leq \sum_{i=n+1}^{m} \|I-T\|^i \leq \sum_{i \geq n+1} \|I-T\|^i = \frac{\|I-T\|^{n+1}}{1-\|I-T\|} \to 0$$

as $n \to \infty$. By the completeness of $\mathcal{L}(\mathcal{X})$ (proved, for example, in Folland's Proposition 5.4) we have $R_n \to R$ in operator norm for some $R \in \mathcal{L}(\mathcal{X})$. By the continuity of multiplication in the operator norm (an immediate consequence of its submultiplicativity), this implies $TR_n \to TR$ and $R_nT \to RT$. However,

$$\begin{aligned} \|I - TR_n\| &= \|I - R_n + (I-T)R_n\| \\ &= \big\|I - \big(I + (I-T) + \cdots + (I-T)^n\big) \\ &\qquad\qquad + \big((I-T) + (I-T)^2 + \cdots + (I-T)^{n+1}\big)\big\| \\ &= \|(I-T)^{n+1}\| \leq \|I-T\|^{n+1} \to 0, \end{aligned}$$

so $TR = \lim_n TR_n = I$, and exactly similarly $RT = I$, so R is a (bounded) right- and left-inverse for T, and so is a bounded inverse.

b. Now suppose that T is invertible in $\mathcal{L}(\mathcal{X})$ and $\|T - S\| \leq \|T^{-1}\|^{-1}$. Then by the submultiplicativity of the norm we have also

$$\|I - ST^{-1}\| = \|(S-T)T^{-1}\| \leq \|S-T\|\|T^{-1}\| < 1,$$

so by part (a) ST^{-1} has a bounded inverse, say R_1. Exactly similarly, $T^{-1}S$ has a bounded inverse, say R_2, and so we have $(ST^{-1})R_1 = S(T^{-1}R_1) = I$ and

$R_2(T^{-1}S) = (R_2T^{-1})S = I$. Therefore we have exhibited both bounded right- and left-inverses for S, so these must agree and S is invertible.

This tells us that for any T in the the subset of invertible operators in $\mathcal{L}(\mathcal{X})$, its neighbourhood $\{S : \ \|S - T\| < \|T^{-1}\|^{-1}\}$ lies in that subset, so the subset is open.

11. [I'm afraid this question really is as long as it looks. Just a couple of remarks to be made here.

Firstly, it takes some diligence to do everything the question asks for:

- For part (a) one must
  - Prove that $\|\cdot\|_{\Lambda_\alpha}$ is a norm (scalar homogeneity; triangle inequality; zero on the zero vector and nowhere else);
  - Prove that Λ_α is complete for this norm: any Cauchy sequence for this norm has a limit function in some sense, that limit function is also in Λ_α, and in fact the sequence converges to the limit function not only in 'some sense', but in the particular sense of $\|\cdot\|_{\Lambda_\alpha}$-norm convergence.
- For part (b) one must
  - Prove when $\alpha < 1$ that λ_α is a sub*space* if Λ_α (closed under addition and scalar multiplication);
  - Prove when $\alpha < 1$ that it is closed for the $\|\cdot\|_{\Lambda_\alpha}$ norm;
  - Prove when $\alpha < 1$ that it is infinite dimensional (that is, exhibit an infinite linearly independent family of members of λ_α);
  - Prove when $\alpha = 1$ that any member of λ_α is a constant function.

Second, this question brings out how important it is to state clearly the order in which we take limits. For example, for the closedness of λ_α in part (b), suppose that we've already proved all of part (a) and that we now have $(f_n)_n \subset \lambda_\alpha$ and $f_n \to f$ in Λ_α. We need to show $f \in \lambda_\alpha$: that is, $\forall\ \varepsilon > 0\ \exists\ \delta > 0$ such that whenever $x \neq y$ in $[0,1]$ and $|x-y| < \delta$ then $|f(x) - f(y)|/|x-y|^\alpha < \varepsilon$.

To this end, given a *fixed* $\varepsilon > 0$ you must *first* pick n large enough that $\|f - f_n\|_{\Lambda_\alpha} < \varepsilon/2$, and now because $f_n \in \lambda_\alpha$ we know that *depending on this* n there is some $\delta > 0$ such that whenever $x \neq y$ in $[0,1]$ and $|x-y| < \delta$ then

$|f_n(x) - f_n(y)|/|x-y|^\alpha < \varepsilon/2$. It follows that for such x, y we have

$$\begin{aligned}\frac{|f(x)-f(y)|}{|x-y|^\alpha} &\leq \frac{|(f-f_n)(x)-(f-f_n)(y)|}{|x-y|^\alpha} + \frac{|f_n(x)-f_n(y)|}{|x-y|^\alpha} \\ &\leq \|f-f_n\|_{\Lambda_\alpha} + \frac{|f_n(x)-f_n(y)|}{|x-y|^\alpha} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\end{aligned}$$

so this δ witnesses the desired inequality for the limit function f and the error tolerance ε.]

Folland Chapter 6

10. Suppose that $f_n, f \in L^p$ and $f_n \to f$ pointwise a.e.

($\Rightarrow$) If $\|f - f_n\|_p \to 0$ then the triangle inequality gives

$$|\|f\|_p - \|f_n\|_p| \leq \|f - f_n\|_p \to 0,$$

as required.

($\Leftarrow$) On the other hand, suppose that $\|f_n\|_p \to \|f\|_p$. We always have

$$|f-f_n|^p \leq (|f|+|f_n|)^p \leq (2\max\{|f|,|f_n|\})^p \leq 2^p \max\{|f|,|f_n|\}^p \leq 2^p(|f|^p+|f_n|^p),$$

so now setting $g_n := 2^p(|f|^p + |f_n|^p)$ we know that this is in L^1 (since we know each of $|f|^p$, $|f_n|^p$ is integrable) and, since $f_n \to f$ pointwise a.e., we have $g_n \to g := 2^{p+1}|f|^p$ pointwise a.e. Finally, we are told that $\|f_n\|_p \to \|f\|_p$, and so

$$\int g_n = 2^p(\|f_n\|_p^p + \|f\|_p^p) \to 2^{p+1}\|f\|_p^p = \int g.$$

Therefore these g_n satisfy the conditions required of the termwise dominating functions in the Generalized Dominated Convergence Theorem (Folland's exercise 2.20), and so that theorem tells us that also

$$\|f - f_n\|_p^p = \int |f - f_n|^p \to \int \lim_{n\to\infty} |f - f_n|^p = \int 0 = 0,$$

as required.

15. Let the situation be as described in the question.

($\Rightarrow$) Suppose $(f_n)_n$ is Cauchy in L^p.

(i) For any fixed $t > 0$ we have

$$\|f_n - f_m\|_p^p = \int |f_n - f_m|^p \geq \int t^p \chi_{\{|f_n - f_m| > t\}} = t^p \mu\{|f_n - f_m| > t\},$$

so now for any $\varepsilon > 0$ we can pick $N \geq 0$ such that $\|f_n - f_m\|_p < \varepsilon^{1/p}/t$, and hence $\mu\{|f_n - f_m| > t\} < \varepsilon$, for all $n, m \geq N$, as required.

(ii) For any $\varepsilon > 0$ there is some $N \geq 0$ such that $\|f_n - f_N\|_p < \varepsilon^{1/p}/2$ for all $n \geq N$. On the other hand, all the functions f_1, f_2, $\ldots$, f_N lie in L^p, and hence all the functions $|f_1|^p$, $|f_2|^p$, $\ldots$, $|f_N|^p$ are integrable. Hence by exercise 3.11 (a) (from assignment 1) we can pick some $\delta > 0$ such that for all $n \leq N$ we have $\int_E |f_n|^p < \varepsilon/(2^p)$ whenever $\mu(E) < \delta$. The above inequalities together with the triangle inequality now imply that for $n \geq N$ we have

$$\int_E |f_n|^p \leq \Big(\|f_n - f_N\|_p + \Big(\int_E |f_N|_p\Big)^{1/p}\Big)^p < (\varepsilon^{1/p}/2 + \varepsilon^{1/p}/2)^p = \varepsilon$$

whenever $\mu(E) < \delta$, so in fact this δ witnesses the uniform integrability of the whole sequence $(|f_n|^p)_n$.

(iii) [This is very similar to the above] First note that for any fixed integrable function g, the Monotone Convergence Theorem implies that

$$\int |g| \chi_{\{|g| > 1/M\}} \uparrow \int |g|$$

as $M \to \infty$, since $|g|\chi_{\{|g|>1/M\}} \uparrow |g|$ pointwise as $M \to \infty$. On the other hand, we clearly have

$$\mu\{|g| > 1/M\} = M \int \frac{1}{M} \chi_{\{|g|>1/M\}} \leq M \int |g| < \infty.$$

Hence for this g and any $\eta > 0$, setting $E := \{|g| > 1/M\}$ for some sufficiently large M gives a set with $\mu(E) < \infty$ and

$$\Big| \int |g| \chi_{\{|f| > 1/M\}} - \int |g| \Big| = \int_{E^c} |g| < \eta.$$

Now, given instead our sequence $(f_n)_n$ in L^p, so that every $|f_n|^p$ is integrable, first choose N as for part (ii) above, and now use the above observation to choose for each $n \leq N$ a set E_n with $\mu(E_n) < \infty$ and

$$\int_{E_n^c} |f_n|^p < \varepsilon/2.$$

Combined with our choice of N and the triangle inequality, this implies that for $n \geq N$ we have

$$\int_{E_N^c} |f_n|^p \leq \Big(\|f - f_n\|_p + \Big(\int_{E_N^c} |f_N|^p \Big)^{1/p} \Big)^p < \varepsilon,$$

so setting $E := \bigcup_{n \leq N} E_n$, this has $\mu(E) < \infty$ (as a finite union of sets of finite measure) and clearly (by monotonicity of the integral) witnesses that

$$\int_{E^c} |f_n|^p < \varepsilon$$

for every n, as required.

($\Leftarrow$) Suppose that (i), (ii) and (iii) hold for a sequence $(f_n)_n$ in L^p and that $\varepsilon > 0$. Then by (iii) we can first choose a measurable set E with $\mu(E) < \infty$ and

$$\int_{E^c} |f_n|^p < \varepsilon/2^p$$

for all n. Next, by (ii) there is some $\delta > 0$ such that

$$\int_A |f_n|^p < \varepsilon/2^p$$

for all n whenever $\mu(A) < \delta$. Finally, setting $A_{nm} := \{|f_n - f_m| \geq (\varepsilon/\mu(E))^{1/p}\}$, by (i) we can choose N such that $\mu(A_{nm}) < \delta$ whenever $n, m \geq N$. Hence for $n, m \geq N$ concatenating the above inequalities and applying the triangle inequality gives

$$\begin{aligned}
\int |f_n - f_m|^p &\leq \int_{E^c} |f_n - f_m|^p + \int_{E \setminus A_{nm}} |f_n - f_m|^p + \int_{A_{nm}} |f_n - f_m|^p \\
&\leq \Big(\Big(\int_{E^c} |f_n|^p \Big)^{1/p} + \Big(\int_{E^c} |f_m|^p \Big)^{1/p} \Big)^p + \int_{E \setminus A_{nm}} |f_n - f_m|^p \\
&\qquad + \Big(\Big(\int_{A_{nm}} |f_n|^p \Big)^{1/p} + \Big(\int_{A_{nm}} |f_m|^p \Big)^{1/p} \Big)^p \\
&\leq \varepsilon + \int_{E \setminus A_{nm}} |(\varepsilon/\mu(E))^{1/p}|^p + \varepsilon \\
&= 3\varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary this completes the proof.

Main class URL: `http://www.math.ucla.edu/~tao/245b.1.09w/`

TA class URL:

`http://www.math.ucla.edu/~timaustin/teaching_245B_W09`

Tim Austin
MS 3931
`timaustin@math.ucla.edu`