

**245B, Winter 2009, Assignment 1:
Notes and selected model answers**

(Model solutions follow question numbers in **bold**.)

Folland Chapter 3

3. a. Our convention for a signed measure ν is simply to define $L^1(\nu) := L^1(\nu^+) \cap L^1(\nu^-)$ in terms of the Jordan Decomposition, and now

$$\begin{aligned} f \in L^1(\nu^+) \cap L^1(\nu^-) &\Leftrightarrow \int |f| d\nu^+ < \infty \ \& \ \int |f| d\nu^- < \infty \\ &\Leftrightarrow \int |f| d\nu^+ + \int |f| d\nu^- < \infty \quad (\because \text{both values } \geq 0) \\ &\Leftrightarrow \int |f| d|\nu| < \infty \\ &\Leftrightarrow f \in L^1(|\nu|). \end{aligned}$$

[Note: although it looks tempting, one can't really 'deduce' from $|\int f| d\nu| < \infty$ that $\int |f| d\nu^+ < \infty$ and $\int |f| d\nu^- < \infty$, because by definition for a *signed* measure we only allow ourselves to write ' $\int |f| d\nu$ ' at all given these two separate finiteness conditions.]

b. It suffices for $f \in L^1(\nu) = L^1(|\nu|)$ to write in terms of the Jordan Decomposition $\nu = \nu^+ - \nu^-$ that

$$\begin{aligned} \left| \int f d\nu \right| &= \left| \int f d\nu^+ - \int f d\nu^- \right| \leq \left| \int f d\nu^+ \right| + \left| \int f d\nu^- \right| \\ &\leq \int |f| d\nu^+ + \int |f| d\nu^- =_{\text{def}} \int |f| d|\nu|. \end{aligned}$$

c. On the one hand, if $X = X_+ \cup X_-$ is a Hahn Decomposition for ν , then

$f := \chi_{X_+} - \chi_{X_-}$ satisfies $|f| \leq 1$ and

$$\left| \int_E f \, d\nu \right| = \nu(E \cap X_+) - \nu(E \cap X_-) =_{\text{def}} \nu^+(E) + \nu^-(E) = |\nu|(E),$$

so the supremum over all such f is certainly at least $|\nu|(E)$. On the other hand, for any measurable f with $|f| \leq 1$ part (b) and monotonicity for the (unsigned) measure $|\nu|$ give

$$\left| \int_E f \, d\nu \right| \leq \int_E |f| \, d|\nu| \leq \int_E \chi_X \, d|\nu| = |\nu|(E),$$

giving the reverse inequality.

7. Be careful for both parts (as in all other evaluations of a supremum) to prove carefully that (a) a claimed ‘supremum’ is an upper bound for the desired set of values, and (b) that no smaller number is an upper bound for that set of values, or, equivalently, that that set of values comes arbitrarily close to the ‘supremum’ from below; and similarly for infima. It turns out in these cases that you can achieve equality for some member of the set of values.

10. Whichever option you choose, be careful to prove both (a) that $\nu \ll \mu$ (i.e., $\mu(E) = 0 \Rightarrow \nu(E) = 0$) and (b) that for some $\varepsilon > 0$, for every $\delta > 0$ there is a measurable set E with $\mu(E) < \delta$ but $\nu(E) \geq \varepsilon$.

11. a. First recall that uniform integrability for a single function is the conclusion of Corollary 3.6 in Folland [note: or prove it yourself, say from the monotone convergence theorem, if you like]. Hence given a finite collection $\{f_{\alpha_1}, f_{\alpha_2}, \dots, f_{\alpha_n}\}$ and $\varepsilon > 0$ we can choose for each $i \leq n$ some $\delta_i > 0$ such that $\left| \int_E f_{\alpha_i} \, d\mu \right| < \varepsilon$ whenever $\mu(E) < \delta_i$. Now the finite minimum $\delta := \min_{1 \leq i \leq n} \delta_i$ is still strictly positive and witnesses the uniform integrability of the whole collection.

b. Given $\varepsilon > 0$, there is some $N \geq 1$ such that $\|f_n - f\|_1 \leq \varepsilon/2$ for $n > N$. On the other hand, by part (a) the finite collection $\{f_1, f_2, \dots, f_N, f\}$ admits some $\delta > 0$ such that $\left| \int_E f_n \, d\mu \right| < \varepsilon/2$ for every $n \leq N$ and also $\left| \int_E f \, d\mu \right| < \varepsilon/2$ whenever $\mu(E) < \delta$. Finally, if $n > N$ and $\mu(E) < \delta$ then

$$\begin{aligned} \left| \int_E f_n \, d\mu \right| &\leq \left| \int_E (f_n - f) \, d\mu \right| + \left| \int_E f \, d\mu \right| \\ &\leq \|f_n - f\|_1 + \left| \int_E f \, d\mu \right| < \varepsilon/2 + \varepsilon/2 = \varepsilon, \end{aligned}$$

so this δ actually witnesses the desired inequality for both $n \leq N$ and $n > N$, and hence the uniform integrability of the whole sequence.

17. This rests on the trick of letting

$$\lambda(E) := \int_E f \, d\mu$$

for $E \in \mathcal{N}$, showing that $\lambda \ll \nu$ and then showing that $g := d\lambda/d\nu$ given by the Radon-Nikodým Theorem has the properties and uniqueness requested. The main warning here is to be careful about where the various different measures are defined: on the one hand, we *cannot* write ‘ $\int_E f \, d\nu$ ’ at any stage because f may not be \mathcal{N} -measurable, and hence not ν -integrable; but on the other, it’s crucial that we adopt the above definition of $\lambda(E)$ *only* for $E \in \mathcal{N}$ (not for all $E \in \mathcal{M}$), since the claim ‘ $\lambda \ll \nu$ ’ can make sense only if these two measures are defined on the same σ -algebra.

Folland Chapter 5

1. Be careful that for scalar multiplication (similarly to addition) you prove continuity letting *both the vector and the scalar vary*; this is what it means to ask for continuity from $K \times \mathcal{X}$ to \mathcal{X} . This, in turn, requires a little care: easiest is to use convergence of sequences and continuity of multiplying two scalars together (both of which you may assume) by writing that if $c_n \rightarrow c$ and $x_n \rightarrow x$ then $\|c_n x_n - cx\| \leq |c_n| \|x_n - x\| + |c_n - c| \|x\| \rightarrow |c| \cdot 0 + 0 \cdot \|x\| = 0$, so $c_n x_n \rightarrow cx$.

5. Note that any point x in the original subspace E is also a limit point in the original subspace by considering the sequence $x_n \equiv x$, so you can consider points of E and $\bar{E} \setminus E$ on an equal footing.

9. a. [Note: here it takes a little thought to unravel exactly how the two given conditions differ.

‘ $f \in C^k([0, 1])$ ’ asserts that f is continuous and has derivatives that are continuous up to order k on the closed interval $[0, 1]$.

‘ $f \in C([0, 1])$, and f is k times continuously differentiable on $(0, 1)$, and $\lim_{h \downarrow 0} f^{(j)}(h)$ exists for all $0 \leq j \leq k$, and $\lim_{h \uparrow 1} f^{(j)}(h)$ exists for all $0 \leq j \leq k$ ’ asserts that f is continuous and has derivatives that are continuous up to order k only on the open interval, together with a convergence property of those derivatives near the end-points. It doesn’t assert, a priori, that f is differentiable *at* the end-points.]

The forward implication is clear, since the continuity of the j^{th} derivative on $[0, 1]$ implies in particular the existence of the end-point limits.

For the reverse implication, we prove by induction on $j = 0, 1, \dots, k$ that f is actually j times differentiable with continuous derivatives on the whole of $[0, 1]$. For the base clause $j = 0$ this asserts simply the continuity of f , which we are given; so now suppose for the recursion clause that we have proved it for all derivatives up to the $(j - 1)^{\text{th}}$ for some $1 \leq j \leq k$.

By inductive hypothesis, the derivatives $f^{(\ell)}(x)$ are defined and continuous for all $x \in [0, 1]$ (in a suitable one-sided sense at 0 and 1) for all $\ell < j$, and by initial assumption we know they have continuous derivatives on $(0, 1)$. Now the mean value theorem tells us that for any $h > 0$ there is some $\xi_h \in (0, h)$ such that

$$\frac{f^{(j-1)}(h) - f^{(j-1)}(0)}{h} = f^{(j)}(\xi_h).$$

Letting $h \downarrow 0$ we must also have $\xi_h \rightarrow 0$, and so by assumption the right-hand side above tends to some limit. The same is therefore true of the left hand-side, but by definition if this tends to some limit then $f^{(j-1)}$ is right-differentiable at 0 with that limit as the derivative. [Note: because we are in the process of arguing that the limit of the left-hand side above exists, we have to argue for all $h \downarrow 0$, not just for some subsequence such as $h = 1/n$, $n \in \mathbb{N}$.] Finally, this also shows that $f^{(j)}(0) = (f^{(j-1)})'(0)$ equals $\lim_{h \downarrow 0} f^{(j)}(h)$, so this derivative is continuous at 0. An exactly similar argument gives the existence and continuity of the derivative at 1, so the induction continues up to $j = k$. This completes the proof.

b. For convenience write

$$\|f\|_k := \sum_{n=0}^k \|f^{(j)}\|_u$$

for the function on $C^k([0, 1])$ under consideration in this question.

First, we observe that it is clearly always a norm, since the triangle inequality and the homogeneity of scalar multiplication are clearly preserved by adding different seminorms, and each $f \mapsto \|f^{(j)}\|_u$ is the uniform norm applied to an image of f under a linear operator and so defines a seminorm. Finally, if $\|f\|_k = 0$ then certainly $\|f\|_u \leq \|f\|_k = 0$, so f must be zero since $\|\cdot\|_u$ by itself is a norm.

It remains to prove completeness, so that $C^k([0, 1])$ is given a Banach space structure by this norm. We will prove this by induction on $k \geq 0$. In the base case $k = 0$ we have $\|f\|_k = \|f\|_u$, the uniform norm, and it is a standard result that this is a complete norm on $C^0([0, 1]) = C([0, 1])$. [Note: you're certainly allowed to just state this.] So suppose for the recursion clause that the completeness of $\|\cdot\|_p$ is known for all $p \leq k - 1$ for some $k \geq 1$, and let us prove the completeness of

$\|\cdot\|_k$. Suppose that $(f_n)_{n \geq 1}$ is a Cauchy sequence in $(C^k([0, 1]), \|\cdot\|_k)$. Then, since $C^{k+1} \subseteq C^k$ and $\|\cdot\|_{k+1} \geq \|\cdot\|_k$, it is certainly a Cauchy sequence in $C^{k-1}([0, 1])$, and so by the inductive hypothesis converges in $\|\cdot\|_{k-1}$ to some $f \in C^{k-1}([0, 1])$. On the other hand, $\|(f_n - f_m)^{(k)}\|_u \leq \|f_n - f_m\|_k$ for all $n, m \geq 1$, so $(f_n^{(k)})_{n \geq 1}$ is a Cauchy sequence in $C([0, 1])$ for the uniform norm, which is known to be complete. Therefore $f_n^{(k)} \rightarrow g$ uniformly for some $g \in C([0, 1])$.

Now, by the convergences we have proved above and the fundamental theorem of calculus we have

$$\begin{aligned} f^{(k-1)}(x) - f^{(k-1)}(0) &= \lim_{n \rightarrow \infty} (f_n^{(k-1)}(x) - f_n^{(k-1)}(0)) \\ &= \lim_{n \rightarrow \infty} \int_0^x f_n^{(k)}(t) dt = \int_0^x \lim_{n \rightarrow \infty} f_n^{(k)}(t) dt = \int_0^x g(t) dt \end{aligned}$$

(where the limit passes inside the integral because the functions converge uniformly [Note: again, this is something you can just state/quote]). Another appeal to the fundamental theorem of calculus finally tells us that $f^{(k-1)}$ must be differentiable with $f^{(k)} = (f^{(k-1)})' = g$, so $f \in C^k([0, 1])$ and

$$\|f_n - f\|_k =_{\text{def}} \|f_n - f\|_{k-1} + \|f_n^{(k)} - g\|_u \rightarrow 0$$

as $n \rightarrow \infty$, and so f is the limit of the Cauchy sequence. Thus the induction continues, and so the proof is complete.

Main class URL: <http://www.math.ucla.edu/~tao/245b.1.09w/>

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