Questions for review of 245A (at start of 245B, W09)

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(Where citations are given, questions taken from N.L. Carothers' *Real Analysis*.)

Measure Theory

(Ch 16 Q 23) Recall the definition of outer measure. Given a subset $E \subseteq \mathbb{R}$, prove that there is a G_δ-set G with $E \subseteq G$ and $m^*(E) = m^*(G)$.

(Ch 16 Q 42) Suppose that $E \subseteq \mathbb{R}$ is Lebesgue measurable with $m(E) = 1$. Show that:

(a) there is a Lebesgue measurable set $F \subseteq E$ with $m(F) = 1/2$.

(b) there is a closed set F consisting entirely of irrationals such that $F \subseteq E$ and $m(F) = 1/2$.

(c) there is a compact set F with empty interior such that $F \subseteq E$ and $m(F) = 1/2$.

(Ch 16 Q 59 & 60) Suppose $E \subseteq \mathbb{R}$, $x, r \in \mathbb{R}$ and $r > 0$. Show that E is Borel (resp. Lebesgue) iff $E + x$ is Borel (resp. Lebesgue) iff $r \cdot E$ is Borel (resp. Lebesgue).

(Ch 17 Q 38) Let $D \subseteq \mathbb{R}$ be measurable and $(f_n)_n$ be a sequence of measurable functions $D \to \mathbb{R}$ converging pointwise a.e. in D to a function $f : D \to \mathbb{R}$. Show that there exist measurable sets $E_1 \subseteq E_2 \subseteq \ldots \subseteq D$ such that $m(D \setminus \bigcup_{n\geq 1} E_n) = 0$ and such that $f_n \to f$ uniformly on each E_k separately.

Integration Theory

(Ch 16 Q 35) If $f : [a, b] \to \mathbb{R}$ is *Riemann* integrable and $f = g$ almost everywhere for Lebesgue measure, does it follow that g is Riemann integrable? What if they agree at all but countably many points? At all but finitely many points?

(Ch 18 Q 17) If $f : \mathbb{R} \to [0, \infty)$ is integrable, prove that the function $x \mapsto$

$\int_{-\infty}^{x} f(x)\,\mathrm{d}x$ is continuous. Prove that in fact more is true: $\forall \varepsilon > 0\ \exists \delta > 0$ such that $\int_E f < \varepsilon$ whenever $m(E) < \delta$.

(Ch 18 Q 41) Let $(f_n)_n$ and f be integrable functions on a σ-finite measure space $(M, \mathcal{B}, \mu)$ such that $f_n \to f$ pointwise a.e.. Prove that $\int |f_n - f|\,\mathrm{d}\mu \to 0$ iff $\int |f_n|\,\mathrm{d}\mu \to \int |f|\,\mathrm{d}\mu$.

Differentiation and integration

(Ch 13 Q 15) Show that $f : [a, b] \to \mathbb{R}$ is both continuous and of bounded variation iff it can be written as the difference of two *strictly* increasing continuous functions.

(Ch 17 Q 33) If $f : (a, b) \to \mathbb{R}$ is differentiable, show that f' is Borel measurable. If f is only differentiable a.e., show that f' is still Lebesgue measurable.

(Ch 20 Q 26) If $g : [0, 1] \to \mathbb{R}$ is Lebesgue integrable and $f : x \mapsto \int_0^x g(x)\,\mathrm{d}x$, prove that the positive and negative variations of f are given by

$$p : x \mapsto \int_0^x \max\{g(x), 0\}\,\mathrm{d}x$$

and

$$n : x \mapsto \int_0^x \min\{g(x), 0\}\,\mathrm{d}x$$

respectively.

Hilbert spaces

Prove that any two separable infinite-dimensional real Hilbert spaces are isomorphic.

(Ch 18 Q 55) Prove the Riemann-Lebesgue lemma: if $f : [0, 2\pi] \to \mathbb{C}$ is in L^1 then $\int_0^{2\pi} f(x)\mathrm{e}^{\mathrm{i}nx}\,\mathrm{d}x \to 0$ as $n \to \infty$.