Math 133 Spring 2009, assignment 6, final question

Offered below is a model solution to the final question from assignment 6 of Math 133, which is not taken from the course textbook (Stein and Shakarchi's *Fourier Analysis*). Note that the solution given here is rather more wordy that would be needed for an actual homework or exam submission, since I have attempted to give some additional motivation for each of the steps that we take.

Question Let $F : \mathbb{R}^2 \to \mathbb{R}$ be continuous and satisfy $|F(x, y)| \leq \frac{1}{(1+x^2)(1+y^2)}$ for all $(x, y) \in \mathbb{R}^2$. Show in detail that

$$\lim_{N\to\infty} \lim_{M\to\infty} \int_{-N}^{N} \int_{-M}^{M} F(x, y)\, dxdy = \lim_{M\to\infty} \lim_{N\to\infty} \int_{-N}^{N} \int_{-M}^{M} F(x, y)\, dxdy,$$

including a proof that all the necessary limits exist.

Remarks **1.** Note that the two sides of the equation we wish to prove involve taking limits in N and in M in the two possible orders, but *not* exchanging the order of the double integral. That can also be done, but it is not the point of the present question, which concerns the behaviour of $\int_{-N}^{N} \int_{-M}^{M} F$ as we enlarge the area of integration to the whole of $\mathbb{R}^2$ in two different ways.

2. There are several ways to begin looking for an answer to this question. First one should observe that the problem falls naturally into two parts: proving that all the required limits exist; and, having done this, showing that the left- and right-hand sides above are equal. Naturally, at first one would attempt to consider these two parts separately. However, it turns out that the same basic estimate underlies the proofs of both parts, and so in the model solution presented below we first derive this estimate, and then show the different ways in which we need to use it.

Solution Let us first fix the notation

$$I_{M,N} := \int_{-N}^{N} \int_{-M}^{M} F(x,y)\, dxdy.$$

Much of our work will be concerned with comparing the values $I_{M,N}$ for different pairs (M,N), so let us first establish a useful estimate for this purpose. Suppose we are given two such pairs (M_1,N_1) and (M_2,N_2). Then we find that

$$\begin{aligned} |I_{M_1,N_1} - I_{M_2,N_2}| &= |I_{M_1,N_1} - I_{M_1,N_2} + I_{M_1,N_2} - I_{M_2,N_2}| \\ &\leq |I_{M_1,N_1} - I_{M_1,N_2}| + |I_{M_1,N_2} - I_{M_2,N_2}| \\ &= \Big| \int_{-\max\{N_1,N_2\}}^{-\min\{N_1,N_2\}} \int_{-M_1}^{M_1} F \Big| + \Big| \int_{\min\{N_1,N_2\}}^{\max\{N_1,N_2\}} \int_{-M_1}^{M_1} F \Big| \\ &\qquad + \Big| \int_{-N_2}^{N_2} \int_{-\max\{M_1,M_2\}}^{-\min\{M_1,M_2\}} F \Big| + \Big| \int_{N_2}^{N_2} \int_{\min\{M_1,M_2\}}^{\max\{M_1,M_2\}} F \Big|. \end{aligned}$$

We would like to find ways to bound this from above by something that is simpler to work with, so now we try some quite rough arguments to this purpose. First, we can bound this expression further from above by moving the absolute value signs inside the integrals, and having done this we are given that we can bound $|F|$ by the explicit expression $\frac{1}{(1+x^2)(1+y^2)}$. This gives

$$\begin{aligned} &|I_{M_1,N_1} - I_{M_2,N_2}| \\ &\leq \int_{-\max\{N_1,N_2\}}^{-\min\{N_1,N_2\}} \int_{-M_1}^{M_1} |F| + \int_{\min\{N_1,N_2\}}^{\max\{N_1,N_2\}} \int_{-M_1}^{M_1} |F| \\ &\qquad + \int_{-N_2}^{N_2} \int_{-\max\{M_1,M_2\}}^{-\min\{M_1,M_2\}} |F| + \int_{N_2}^{N_2} \int_{\min\{M_1,M_2\}}^{\max\{M_1,M_2\}} |F| \\ &\leq \int_{-\max\{N_1,N_2\}}^{-\min\{N_1,N_2\}} \int_{-M_1}^{M_1} \frac{1}{(1+x^2)(1+y^2)} + \int_{\min\{N_1,N_2\}}^{\max\{N_1,N_2\}} \int_{-M_1}^{M_1} \frac{1}{(1+x^2)(1+y^2)} \\ &\qquad + \int_{-N_2}^{N_2} \int_{-\max\{M_1,M_2\}}^{-\min\{M_1,M_2\}} \frac{1}{(1+x^2)(1+y^2)} + \int_{-N_2}^{N_2} \int_{\min\{M_1,M_2\}}^{\max\{M_1,M_2\}} \frac{1}{(1+x^2)(1+y^2)}. \end{aligned}$$

Now we notice that the function $\frac{1}{(1+x^2)(1+y^2)} = \frac{1}{1+x^2} \cdot \frac{1}{1+y^2}$ breaks into a product of functions each depending on only one of the coordinates, and so for any intervals $[A_1,A_2]$ and $[B_1,B_2]$ in $\mathbb{R}$ we have

$$\int_{B_1}^{B_2} \int_{A_1}^{A_2} \frac{1}{(1+x^2)(1+y^2)}\, dxdy = \Big(\int_{A_1}^{A_2} \frac{1}{1+x^2} \Big) \Big(\int_{B_1}^{B_2} \frac{1}{1+y^2} \Big),$$

and so we can re-write the above upper bound as

$$\begin{aligned}
&|I_{M_1,N_1} - I_{M_2,N_2}| \\
&\leq \Big(\int_{-M_1}^{M_1} \frac{1}{1+x^2}\Big)\Big(\int_{-\max\{N_1,N_2\}}^{-\min\{N_1,N_2\}} \frac{1}{1+y^2}\Big) + \Big(\int_{-M_1}^{M_1} \frac{1}{1+x^2}\Big)\Big(\int_{\min\{N_1,N_2\}}^{\max\{N_1,N_2\}} \frac{1}{1+y^2}\Big) \\
&\qquad + \Big(\int_{-\max\{M_1,M_2\}}^{-\min\{M_1,M_2\}} \frac{1}{1+x^2}\Big)\Big(\int_{-N_2}^{N_2} \frac{1}{1+y^2}\Big) + \Big(\int_{\min\{M_1,M_2\}}^{\max\{M_1,M_2\}} \frac{1}{1+x^2}\Big)\Big(\int_{-N_2}^{N_2} \frac{1}{1+y^2}\Big).
\end{aligned}$$

This is the basic estimate that will underly the rest of the proof, together with the fact that $\int_{-\infty}^{\infty} \frac{1}{1+x^2}$ is some finite positive constant, say C (in fact $C = \arctan(+\infty) - \arctan(-\infty) = \pi$).

We will now show how to complete the solution given this estimate. Recall that we must prove two separate things: firstly that all the desired limits exist, and secondly that the desired left- and right-hand expressions are equal.

First part We give a proof that $\lim_{N\to\infty} \lim_{M\to\infty} I_{M,N}$ exists, the argument for the other side being very similar. Let us first write out carefully what the existence of this double limit means: we need to show that

- for every $N \geq 1$ the values $I_{M,N}$ tend to a limit in $\mathbb{R}$, say J_N, as $M \to \infty$,
- and that as $N \to \infty$, these limiting values J_N in turn tend to some limit J_∞.

Thus the proof that our double limit exists breaks naturally into two further stages, corresponding to the two parts of this assertion.

First consider the numbers $I_{M,N}$ for a fixed value of N as $M \to \infty$. We need to show that they tend to a limit in $\mathbb{R}$. Since the function F is fairly arbitrary, we do not have any particular candidate value for this limit, and so to show that a limit exists we will instead prove the equivalent assertion that the sequence of numbers $(I_{M,N})_{M\geq 1}$ is Cauchy.

Thus, unraveling the desired result a little further, we must show that for any $N \geq 1$ and $\varepsilon > 0$ there is some $M_0(N,\varepsilon) \geq 1$ such that

$$M_1, M_2 \geq M_0(N,\varepsilon) \qquad \Rightarrow \qquad |I_{M_1,N} - I_{M_2,N}| < \varepsilon.$$

However, in this case taking $N_1 = N_2 = N$ in the basic estimate we find that

$\min\{N_1, N_2\} = \max\{N_1, N_2\}$ and so that estimate simplifies to

$$\begin{aligned}
&|I_{M_1,N} - I_{M_2,N}| \\
&\leq \int_{-N}^{N} \frac{1}{1+y^2} \int_{-\max\{M_1,M_2\}}^{-\min\{M_1,M_2\}} \frac{1}{1+x^2} + \int_{-N}^{N} \frac{1}{1+y^2} \int_{\min\{M_1,M_2\}}^{\max\{M_1,M_2\}} \frac{1}{1+x^2} \\
&\leq \int_{-N}^{N} \frac{1}{1+y^2} \int_{-\infty}^{-\min\{M_1,M_2\}} \frac{1}{1+x^2} + \int_{-N}^{N} \frac{1}{1+y^2} \int_{\min\{M_1,M_2\}}^{\infty} \frac{1}{1+x^2} \\
&\leq C\Big(\int_{-\infty}^{-\min\{M_1,M_2\}} \frac{1}{1+x^2} + \int_{\min\{M_1,M_2\}}^{\infty} \frac{1}{1+x^2}\Big),
\end{aligned}$$

where for the second inequality we use the fact that $\int_{-N}^{N} \frac{1}{1+y^2} \leq \int_{-\infty}^{\infty} \frac{1}{1+y^2} = C$ for any N, since $\frac{1}{1+y^2} > 0$.

Now notice that $\int_{-L}^{L} \frac{1}{1+x^2} \longrightarrow \int_{-\infty}^{\infty} \frac{1}{1+x^2}$ as $L \longrightarrow \infty$, and hence that

$$\int_{-\infty}^{-L} \frac{1}{1+x^2} + \int_{L}^{\infty} \frac{1}{1+x^2} = \int_{\infty}^{\infty} \frac{1}{1+x^2} - \int_{-L}^{L} \frac{1}{1+x^2} \longrightarrow 0$$

as $L \longrightarrow \infty$. It follows that for any $\varepsilon > 0$ there is some $M_0(\varepsilon) > 0$ such that provided $M_1, M_2 \geq M_0(\varepsilon)$ (and hence also $\min\{M_1, M_2\} \geq M_0(\varepsilon)$) then we have

$$\int_{-\infty}^{-\min\{M_1,M_2\}} \frac{1}{1+x^2} + \int_{\min\{M_1,M_2\}}^{\infty} \frac{1}{1+x^2} \leq \int_{-\infty}^{-M_0(\varepsilon)} \frac{1}{1+x^2} + \int_{M_0(\varepsilon)}^{\infty} \frac{1}{1+x^2} < \varepsilon/C$$

and hence

$$|I_{M_1,N} - I_{M_2,N}| < \varepsilon.$$

This gives that $I_{M,N}$ is Cauchy in M for fixed N, so we may define J_N to be its limiting value. We must now show that these in turn form a Cauchy sequence as $N \longrightarrow \infty$, and so converge to a limit J_∞ of their own. To this end we will in fact show that

for any $\varepsilon > 0$ there is some $N_0(\varepsilon) > 0$ such that

$$N_1, N_2 \geq N_0(\varepsilon) \qquad \Rightarrow \qquad |I_{M,N_1} - I_{M,N_2}| \leq \varepsilon\ \forall M > 0, \quad (1)$$

where the crucial feature is that the choice of $N_0(\varepsilon)$ witnessing this bound does not depend on M. Given this, we may let $M \longrightarrow \infty$ to obtain that for any $\varepsilon > 0$ there is some $N_0(\varepsilon) > 0$ such that

$$N_1, N_2 \geq N_0(\varepsilon) \qquad \Rightarrow \qquad |J_{N_1} - J_{N_2}| \leq \varepsilon,$$

as required for a Cauchy sequence.

To prove (1) we return to our basic estimate, this time taking $M_1 = M_2 = M$ and obtaining a similar simplification to that above to see that

$$|I_{M_1,N} - I_{M_2,N}| \leq C\Big(\int_{-\infty}^{-\min\{N_1,N_2\}} \frac{1}{1+y^2} + \int_{\min\{N_1,N_2\}}^{\infty} \frac{1}{1+y^2}\Big)$$

independently of M. Now arguing precisely as above we see that this can be made less than ε by choosing N_1 and N_2 both separately large enough, so giving the proof of (1).

This shows that $\lim_{N\to\infty} \lim_{M\to\infty} I_{M,N} = J_\infty$ exists; exactly similarly we can prove that for each fixed M the sequence $I_{M,N}$ is Cauchy as $N \to \infty$, so converges to some limit K_M, and then that these form a Cauchy sequence as $M \to \infty$, and so themselves have a limit $K_\infty = \lim_{M\to\infty} \lim_{N\to\infty} I_{M,N}$.

Second part It only remains to prove that $J_\infty = K_\infty$. This must clearly hold if we prove that $|J_\infty - K_\infty| \leq \varepsilon$ for every $\varepsilon > 0$. By taking limits we deduce that this will hold if we show that for every $\varepsilon > 0$ we have $|J_N - K_M| \leq \varepsilon$ for all sufficiently large N and M, and now by taking further limits we find that this, in turn, will follow if

> for every $\varepsilon > 0$, for all sufficiently large N and M it is the case that for all sufficiently M' and N' we have
>
> $$|I_{M',N} - I_{M,N'}| \leq \varepsilon. \tag{2}$$

This, finally, will follow from one last application of the basic estimate. Indeed, once again we can choose L so large that

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} - \int_{-L}^{L} \frac{1}{1+x^2} < \varepsilon/(2C),$$

and now provided only that we choose $M, N, M', N' \geq L$, and writing

$$\begin{aligned} \Big(\int_{-M'}^{M'} \frac{1}{1+x^2}\Big)\Big(\int_{-\max\{N,N'\}}^{-\min\{N,N'\}} \frac{1}{1+y^2}\Big) \;&\leq\; \Big(\int_{-\infty}^{\infty} \frac{1}{1+x^2}\Big)\Big(\int_{-\infty}^{-L} \frac{1}{1+y^2}\Big) \\ &=\; C\int_{-\infty}^{-L} \frac{1}{1+y^2} \end{aligned}$$

and similarly for the other three terms on the right-hand side of the basic esimate, we obtain

$$\begin{aligned}&|I_{M',N} - I_{M,N'}|\\ &\leq C\int_{-\infty}^{-L}\frac{1}{1+y^2}+C\int_{L}^{\infty}\frac{1}{1+y^2}+C\int_{-\infty}^{-L}\frac{1}{1+x^2}+C\int_{L}^{\infty}\frac{1}{1+x^2} < 2C(\varepsilon/2C) = \varepsilon\end{aligned}$$

whenever $M, M' > L$ and $N, N' > L$. This completes the proof of (2), and hence also the proof that the two double limits agree in this case. $\square$

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