Analysis on Metric Spaces

Summer School, Lake Arrowhead *

September 09th - September 14th 2007

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*supported by NSF grant DMS 0701302

Contents

1 Metric and geometric quasiconformality in Ahlfors regular Loewner spaces

after Jeremy T. Tyson
A summary written by Jonas Azzam

Abstract

Here we describe Tyson's theorem on the equivalence of geometric quasiconformality, local-quasisymmetry, and a variant of metric quasiconformality on domains of metric spaces and give a summary of the main points and techniques of its proof.

1.1 Preliminaries

Let (X, d) and (Y, d') be metric spaces. We denote $d(x, y)$ and $d'(x, y)$ by $|x - y|$ when there is no confusion about which space and metric we are concerned with. A map $f : X \to Y$ is said to be *quasisymmetric* or η-*quasisymmetric* if, for all $x, y, z \in X$ and for any $t > 0$,

$$\text{whenever } |x-y| \leq t|y-z|, \text{ we have } |f(x)-f(y)| \leq \eta(t)|f(y)-f(z)| \quad (1)$$

for some homeomorphism $\eta : [0, \infty) \to [0, \infty)$. We say that f is *locally quasisymmetric* if there exists a homeomorphism η such that every point in X has a neighborhood in which (1) holds for x, y, z in this neighborhood.

We write $A \lesssim B$ if $A \leq CB$, and $A \sim B$ if $A/C \leq B \leq CA$ for some constant $C \geq 1$. We refer to the constant C as the *implied constant* in the inequality.

A Borel measure μ is *(Ahlfors-David) Q-regular* if for any open ball $B(x, r)$ centered at x of radius r in X,

$$\mu(\overline{B}(x, r)) \sim r^Q. \quad (2)$$

We say that X is *Q-regular* if there exists such a measure on X. A measure μ is *locally Q-regular* if the above condition holds for all x for sufficiently small $r > 0$.

Let X be a metric space with a locally finite Borel measure μ. For a collection of curves Γ and a number p, we define the *p-modulus of* Γ to be

$$\mathrm{Mod}_p\Gamma = \inf_{\rho} \int_X \rho^p d\mu$$

where the infimum is taken over all Borel functions $\rho : X \to [0, \infty]$ such that $\int_\gamma \rho \geq 1$ (by a *curve*, we will mean a continuous nonconstant function $\gamma : I \to X$ for some interval (possibly noncompact) $I \subseteq \mathbb{R}$). Such a ρ satisfying these conditions we call *admissible* for Γ.

The modulus satisfies the following properties:

(i) $\mathrm{Mod}_p\emptyset = 0$;

(ii) $\Gamma \subseteq \Gamma'$ implies $\mathrm{Mod}_p\Gamma \leq \mathrm{Mod}_p\Gamma'$;

(iiii) Mod_p is countably subadditive;

(iv) $\mathrm{Mod}_p\Gamma \leq \mathrm{Mod}_p\Gamma'$ whenever every curve in Γ has a curve in Γ' as a subcurve;

(v) If $1 < p < \infty$ and Γ_n is an increasing sequence of curve families whose union is Γ, then $\lim_{n\to\infty} \mathrm{Mod}_p\Gamma_n = \mathrm{Mod}_p\Gamma$.

With X as above, X is said to be a *p-Loewner space* if there exists a nonincreasing positive function $\varphi : (0, \infty) \to (0, \infty)$ such that

$$\mathrm{Mod}_p(E, F) \geq \varphi(t) > 0$$

whenever E and F are disjoint nondegenerate (i.e. they are not singleton points) continua in X such that

$$\mathrm{dist}(E, F) \leq t \min\{\mathrm{diam}E, \mathrm{diam}F\},$$

where (E, F) denotes the collection of curves connecting E to F. We call φ a *Loewner function of X.*

A function $f : X \to Y$ between two metric measure spaces (each having a locally finite Bore measure) is said to be *geometrically K- quasiconformal* if, for every collection of curves Γ,

$$\frac{1}{K}\mathrm{Mod}_p\Gamma \leq \mathrm{Mod}_p \leq K\mathrm{Mod}_p\Gamma.$$

Define, for $r > 0$,

$$L_f(x, r) = \sup\{|f(x) - f(y)| : |x - y| \le r\}$$

and

$$\ell_f(x, r) = \inf\{|f(x) - f(y) : |x - y| \ge r\}.$$

We say that f is *metrically H-quasiconformal* if there exists $H \ge 1$ such that

$$\limsup_{r \to 0} \frac{L_f(x, r)}{\ell_f(x, r)} \le H$$

for all $x \in X$.

1.2 Main theorem and discussion

The problem considered in this paper is the equivalence of quasiconformality and quasisymmetry in metric spaces. In particular, the author has proved earlier (see [3], theorem 1.4) that geometric quasiconformality and quasisymmetry were equivalent for maps f between Q-regular metric measure spaces. Furthermore, we know that metric quasiconformality is equivalent to local-quasisymmetry for homeomorphisms between Q-regular Loewner spaces (see [2]), and therefore that quasisymmetry and metric and geometric quasiconformality are equivalent for homeomorphisms between Q-regular Loewner spaces.

The author raises the question of whether one can generalize these results to maps between subdomains of these spaces, i.e. $f : U \to U'$ where $U \subseteq X$ and $U' \subseteq Y$, where U and U' need not be Loewner or Q-regular. The result is true for $\mathbb{R}^n$ and we ask how far we can extend it to arbitrary metric spaces. In this context, the author proves the following result:

Theorem 1. *Let (X, d, μ) and (Y, d', ν) be locally compact, Q-regular spaces with $Q > 1$, where X is Loewner and Y is locally linearly connected. Let $U \subseteq X$ and $U' \subseteq Y$ be open sets and $f : U \to U'$ a homeomorphism. Then the following are equivalent:*

(i) there exists $t > 1$ and $H < \infty$ such that

$$\liminf \frac{L_f(x, tr)}{\ell_f(x, r)} \le H$$

for all $x \in X$;

(ii) for each $t > 1$, there exists $H(t) < \infty$ such that

$$\limsup_{r\to 0} \frac{L_f(x,tr)}{\ell_f(x,r)} \leq H(t)$$

for all $x \in X$;

(iii) there exists a $K < \infty$ such that f is geometrically K-quasiconformal;

(iv) there exists $\eta : [0,\infty) \to [0,\infty)$ such that f is locally η-quasisymmetric.

A metric space Y is *C-locally linearly connected* if there exits $C \geq 1$ such that, for every point x and every $r > 0$, any two points in $B(x,r)$ can be connected by a curve lying in $B(x,Cr)$ and every two points in $B(x,r)^c$ may be connected by a curve in $B(x,r/C)$. Note that this is a weaker condition than Y being Loewner, that is, Loewner spaces are linearly locally connected (see [1], theorem 8.23). Note also that the theorem does not achieve equivalence with metric quasiconformality, but a weaker variant described by the conditions (i) and (ii).

1.3 Details and techniques of proof

1.3.1 Outline

We briefly outline here the steps for proving the main result to put the later details in context:

1. We first develop a generalized form of modulus, $\mathrm{Mod}^{\bullet}_{p,\sigma}$, which is constructed depending on data σ that we specify. We will sketch the construction of this in the section below.

2. For locally Q-regular metric spaces, this new modulus is comparable to the traditional modulus.

3. For this new modulus, if condition (i) is satisfied by a homeomorphism f, then $\mathrm{Mod}^{\bullet}_{p,\sigma}\Gamma \leq \mathrm{Mod}^{\bullet}_{p,\sigma'} f\Gamma$, where the data σ and σ' depend on the constants t and H. Pulling this together, we get that

$$\mathrm{Mod}_p\Gamma \lesssim \quad \mathrm{mod}\ {}^{\bullet}_{p,\sigma}\Gamma \leq \mathrm{Mod}^{\bullet}_{p,\sigma'} f\Gamma \lesssim \mathrm{Mod}_p f\Gamma. \tag{3}$$

   Hence, this shows (i) implies "half" of (iii), that is, one half of the inequality for quasiconformality.

4. We then show that if this half of (iii) is satisfied, then this implies quasisymmetry.

In summary, we get (i) implies half of (iii) which in turn implies (iv). It is not hard to show (iv) implies (ii) (the only assumption needed here on X and Y is that Y is locally C-linearly locally connected), (ii) implies (i) trivially, and since the inverse of a quasisymmetric map f is also quasisymmetric, the above discussion implies the reverse inequality in (3), so (iv) implies (iii), so all statements are equivalent. Since the bulk of the proof lies in the first three steps, we will discuss those below and mention the result needed for step 4 in the end.

1.3.2 Quasiround sets and generalized packing measure

Here we introduce quasiround sets and describe how to construct the generalized packing measure needed to define the generalized modulus. The author presents the construction in a more general setting, working with quasiround balls instead of restricting all attention to just balls. A normal ball under the maps we will consider may be morphed to some degree, so by weakening our definition of roundness these balls will fall within the same class of objects under these maps: quasiround sets will be mapped to quasiround sets. We can then develop the tools we have defined for usual balls (doubling set functions, covering lemmas, et cetera) for this larger class.

Let $A \subseteq X$ be a closed set. If $k > 1$, we say A is *k-quasiround* if there exists $x \in X$ such that

$$B(x, r_1) \subseteq A \subseteq B(x, r_2)$$

where $0 < r_1 < r_2 < kr_1$. We call r_1 and r_2 the inner and outer radius respectively. We say that a set A is *1-quasiround* if $A = B(x, r)$ for some $x \in X$ and $r > 0$. We denote $\mathcal{B}_k(X)$ to be the set of all quasiround sets, $\mathcal{B}_1(X)$ to be the set of all balls, and $\mathcal{B}(X)$ to be all quasiround sets. Two sets A and A' are an *ℓ-quasiring* if there exist $r_1 < r_2 < \ell r_1 < \infty$ and $x \in X$ such that

$$\overline{B}(x, r_1) \subseteq A \subseteq A' \subseteq \overline{B}(x, r_2).$$

These are generalizations of balls and concentric balls, allowing the boundary of the ball to vary by a factor.

Let $E \subseteq X$ and $\mathcal{V}$ a collection of quasiround subsets of X. We say that $\mathcal{V}$ is a *fine* (or *Vitali*) cover of E if every point in E is contained in a quasiround set in $\mathcal{V}$ of arbitrarily small outer radius. We say a subcollection $\mathcal{C} \subseteq \mathcal{V}$ is $\mathcal{V}$*-full* if there is a function $\Delta : E \to (0, \infty)$ such that every element of $\mathcal{V}$ centered at x with outer radius at most $\Delta(x)$ is in $\mathcal{C}$. Two fine collections $\mathcal{V}$ and $\mathcal{V}$ are an ℓ*-admissible pair* if there exists a map $A \mapsto A'$ from $\mathcal{V}$ to $\mathcal{V}'$ such that (A, A') is an ℓ-quasiring. This is the generalization of doubling of balls, so for example, $\mathcal{B}_1(X)$ is admissible with itself via the doubling map $B(x, r) \mapsto B(x, 2r)$.

For $\mathcal{V} \subseteq \mathcal{B}_k(X)$, $k \geq 1$, a fine cover of X, $\mathcal{C} \subseteq \mathcal{V}$ a subcollection, a set function $\psi : \mathcal{V} \to [0, \infty]$, and $\delta \in (0, \infty]$, define the *variation*

$$V(\psi, \mathcal{C}) = \sup \sum_i \psi(A_i)$$

where the supremum is taken over all finite or countable collections $\{A_i\} \subseteq \mathcal{C}$ of pairwise disjoint elements. Then, for a set $E \subseteq X$, define

$$\Psi^\bullet_{k,\mathcal{V}}(E) = \inf_{\mathcal{C}} V(\psi, \mathcal{C})$$

where the infimum is over all $\mathcal{V}$-fine covers $\mathcal{C}$ of E. This turns out to be a Borel regular measure on X.

Furthermore, given the same ψ and $\mathcal{V}$ as above, we can develop a generalized notion of length. For a curve $\gamma : I \to X$, define

$$\Psi - \text{length}_{k,\mathcal{V}} = \sup_{\gamma_0} \lim_{\delta \to 0} \inf \sum \psi(A_i)$$

where the supremum is taken over subcurves of γ by restricting to compact subintervals of I, and the infimum is taken over finite or countable covers $\{A_i\} \subseteq \mathcal{V}$ of the image of γ_0 with outer radii at most δ.

For an ℓ-admissible pair $(\mathcal{V}, \mathcal{V}')$, we say a set function $\psi : \mathcal{V} \cup \mathcal{V}' \to [0, \infty]$ is *locally M-blanketed* for some $M > 0$ if the collection of sets $A \in \mathcal{V}$ for which

$$\psi(A') \leq M\psi(A)$$

is a $\mathcal{V}$-full cover of X.

1.3.3 The generalized modulus and its role

Now we may define the generalized modulus and begin the steps of the proof of the main theorem. Let X be a metric space. Let $\ell > k \geq 1$, $M < \infty$, and

$\mathcal{V}, \mathcal{V}' \subseteq \mathcal{B}_k(X)$ be fine covers of X which form an ℓ-admissible pair. Let Γ be a family of curves. We define the *(packing-type) generalized p-modulus of* Γ with respect to the data $\sigma = (k, \ell, M)$ to be

$$\mathrm{Mod}^{\bullet}_{p,\sigma}\Gamma = \inf \Psi^{\bullet}_{p,k,\mathcal{V}}(X),$$

where the infimum is taken over all generalized packing measures $\Psi_{p,k,\mathcal{V}}$ generated by set functions ψ^p where $\psi : \mathcal{V} \cup \mathcal{V}' \to [0, \infty]$ is a locally blanketed M-function such that $\Psi - \mathrm{length}(\gamma) \geq 1$ for all $\gamma \in \Gamma$. Such a ψ satisfying these conditions is said to be *admissible* for Γ.

We may prove that this and the original modulus are comparable via the following lemma. First, for a metric space X, we define the *constant of uniform perfectness* to be the constant $c > 0$ such that $\mathrm{diam}\overline{B}(x, r) \geq 2cr$ for every $x \in X$ and $r > 0$, and X is *uniformly perfect* if such a constant exists.

Lemma 2. *Let (X, μ) be a locally Q-regular metric measure space with $Q > 1$. Let $t > \ell > 1$ and $M < \infty$. Then*

$$\mathrm{Mod}_Q\Gamma \lesssim \mathrm{Mod}^{\bullet}_{Q,(1,\ell,M)}\Gamma$$

for any family of curves Γ and fine collections $\mathcal{V} \subseteq \mathcal{B}_1(X)$ with $\mathcal{V}' = t\mathcal{V}$. If in addition X is locally compact, then, for fixed $k \geq 1$,

$$\mathrm{Mod}^{\bullet}_{Q,(k,\ell,M)}\Gamma \lesssim \mathrm{Mod}_Q\Gamma$$

for all $\ell > k$ and for any $M \geq \ell/c$, where the implied constants depends only on k, Q, and the implied constants in equation (2), and c is the constant of uniform perfectness for X.

Note that Q-regular spaces are uniformly perfect. Both inequalities are proven with a similar motif in mind. In the first inequality, we let $\psi : \mathcal{V} \cup \mathcal{V}'$ be any admissible function for Γ with respect to the generalized modulus and use it to construct a particular admissible Borel function ρ for the classical modulus that will help us achieve the bounds we need to prove the inequality. Proving the second inequality follows similarly, by picking any ρ admissible for the classical modulus and constructing a ψ admissible for the generalized modulus.

Next, there is the task of showing step 3 in the proof.

Lemma 3. *Let $f : X \to Y$ be a homeomorphism between locally compact metric spaces satisfying condition (i) in the main theorem. Let $\ell > t$, $\ell' = k' > H$, and $M < \infty$. Let $\mathcal{V}$ be the fine collection of balls B such that the image of the ℓ-ring (B, tB) under f is an ℓ'-quasiring. Then we have that, for data σ and σ' depending on t and H,*

$$\mathrm{Mod}^{\bullet}_{p,(1,\ell,M)}\Gamma \lesssim \mathrm{Mod}^{\bullet}_{p,(k',\ell',M)} f\Gamma$$

where $\mathrm{Mod}^{\bullet}_{p,(k',\ell',M)}$ is computed relative to the ℓ'-admissible pair $(f(\mathcal{V}), f(t\mathcal{V}))$

Condition (i) of the theorem ensures that $\mathcal{V}$ is a fine cover of X. We outline the main points of the proof: let $\psi : f(\mathcal{V}) \cup f(t\mathcal{V})$ be an admissible locally M-blanketed function for the generalized modulus of $f\Gamma$, and define $\varphi : \mathcal{V} \to [0, \infty]$ as

$$\varphi(A) := \psi(fA)$$

for $A \in \mathcal{V} \cup t\mathcal{V}$. Then φ will be locally M-blanketed. Next, we show that

1. Φ-length$_{1,t\mathcal{V}} \geq \Psi$-length$_{\ell',f(t\mathcal{V})}$ for all $\gamma \in \Gamma$;
2. $\Phi^{\bullet}_{p,1,\mathcal{V}}(E) \leq \Psi^{\bullet}_{p,\ell',f\mathcal{V}}(E)$ for every Borel set $E \subseteq X$.

Then φ is admissible for the generalized modulus of Γ by (1) and (2) finishes the proof by letting $E = X$ and taking the infimum of both sides.

The proof is complete once we mention this last lemma, which completes step 4 of the proof.

Lemma 4. *Let (X, μ) be a Q-regular Loewner space and let (Y, ν) be a Q-regular C-locally linearly connected space with $Q > 1$. Let $f : U \to U'$ be a homeomorphism between open subsets of X and Y respectively which satisfies $\mathrm{Mod}_Q\Gamma \lesssim \mathrm{Mod}_Q f\Gamma$ for all curve families Γ in U. Then f is locally quasisymmetric.*

References

[1] J. Heinonen, *Lectures on analysis on metric spaces.* Springer, New York, NY, 2001.

[2] J. Heinonen and P. Koskela, *Quasiconformal maps in metric spaces with controlled geometry*, Acta Math. **181** (1998), 1-61.

[3] J. T. Tyson, *Metric and geometric quasiconformality in Ahlfors regular Loewner spaces*, Conform. Geom. Dyn. **5** (2001), 21-73.

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2 Quasi-symmetry and 2-spheres

after Bonk and Kleiner
A summary written by Michael Bateman

Abstract

Let Z be a 2-regular metric space homeomorphic to S^2. Then Z is quasimobius to S^2 iff Z is linearly locally connected.

2.1 Introduction and definitions

We sketch a proof of one of the main theorems of *Quasisymmetric parametrizations of two dimensional metric spheres* by Bonk and Kleiner, stated here in the abstract. The theorem proved there uses "quasisymmetric" instead of "quasimobius", and "linearly locally contractible" instead of "linearly locally connected". These notions are equivalent in our setting, and we focus on the version stated here. A metric space Z is λ-*linearly locally connected* (λ -LLC) if x, y can be connected inside λB whenever x, y are in the same ball B, and if x, y can be connected in $Z \backslash B$ whenever $x, y \in Z \backslash \lambda B$. If $x_1, x_2, x_3, x_4 \in X$ are distinct, define the *modified cross-ratio* as

$$\langle x_1, x_2, x_3, x_4 \rangle = \frac{\min(d(x_1, x_3), d(x_2, x_4))}{\min(d(x_1, x_4), d(x_2, x_3))}. \tag{1}$$

A homeomorphism $f : Z_1 \to Z_2$ is η-quasimobius if $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ is a homeomorphism, $\eta(t) \to 0$ as $t \to 0$, and

$$\langle f(x_1), f(x_2), f(x_3), f(x_4) \rangle \leq \eta\left(\langle x_1, x_2, x_3, x_4 \rangle\right). \tag{2}$$

A *continuum* is a connected compact set. If E, F are continua, define their relative distance to be

$$\Delta(E, F) = \frac{dist(E, F)}{\min(diam(E), diam(F))}. \tag{3}$$

Proposition 1. *Let Z be LLC. Then for all $\epsilon > 0$, we have*

A. If $\langle x_1, x_2, x_3, x_4 \rangle$ is small enough, then there are continua E and F such that $x_1, x_3 \in E$ and $x_2, x_4 \in F$, and $\Delta(E, F) \geq \frac{1}{\epsilon}$.

B. If there are continua E, F such that $x_1, x_3 \in E$ and $x_2, x_4 \in F$ and $\Delta(E, F)$ is small enough , then $\langle x_1, x_2, x_3, x_4 \rangle < \epsilon$.

Proof. We include the proof of B, which is simple:

$$\langle x_1, x_2, x_3, x_4 \rangle \leq \frac{\min(diam(E), diam(F))}{dist(E,F)} = \frac{1}{\Delta(E,F)}. \tag{4}$$

□

We end this section by remarking that if $D_k \subseteq X$, with X, Y compact, and $f_k : D_k \to Y$ is a sequence of η quasimobius functions (with the same control function!) where the D_k's become arbitrarily dense in the limit, then we may extract a convergent subsequence by using a standard diagonalization argument, as long as we assume that there are $x_1^k, x_2^k x_3^k$ and $C \geq> 0$ such that

$$d(x_i^k, x_j^k) > \frac{diam(X)}{C} \text{ and } d(f(x_i^k), f(x_j^k)) > \frac{diam(Y)}{C}. \tag{5}$$

This ensures that the f_k are equicontinuous.

2.2 Approximating metric spaces by graphs

Let G be a graph with vertex set V. We write $u \sim v$ to mean that there is an edge connecting u and v. Let $\mathcal{A} = (G, p, r, \mathcal{U})$ represent a graph G, a function $r : V \to (0, \infty)$, a function $p : V \to Z$, and an open cover $\mathcal{U} = \{U_v\}_{v \in V}$ of Z. $K \in \mathbb{N}^+$, define the K-star of V to be $St_K(v) = \bigcup_{k(v,w)<K} U_w$, where k denotes combinatorial distance in the graph. Let $N_s(v) = \{w \in V : k(w,v) \leq s\}$. Define $mesh(\mathcal{A}) = sup_{v \in V} r(v)$. If $K \in \mathbb{N}^+$, we say $\mathcal{A}$ is a K-approximation of Z if

1. The degree of every vertex $v \in V$ is bounded by K.
2. $B(p(v), r(v)) \subseteq U_v \subseteq B(p(v), Kr(v))$, and $N_{\frac{r(v)}{K}} \subseteq St_K(v)$ for all v.
3. If $v \sim w$, then $U_v \cap u_w \neq \emptyset$, and $r(v) \leq Kr(w)$. If $U_v \cap u_w \neq \emptyset$, then $k(v,w) < K$.
4. If $x, y \in U_v$, then there is a curve γ connecting x and y in $St_K(v)$.

Conditions 1 and 3 give an upper bound on the number of neighborhoods (or K-stars) that can cover the same point, and Condition 5 is similar to the LLC condition. The map p tells us the location of v in Z.

Proposition 2. *If Z is LLC, doubling and homeomorphic to S^2, then there is a K-approximation $\mathcal{A}$ of Z with arbitrarily small mesh size such that the graph G of $\mathcal{A}$ is the 1-skeleton of a triangulation of S^2.*

This is proved by taking a net in the space, connecting points that are close together with the LLC condition, and verifying the conditions for approximations. Adding a few extra points and edges makes it into a triangulation, which is what we want. Now given any graph that is a triangulation of S^2, we can realize it as a circle packing with circles having centers at the vertices, by a result of Andreev-Koebe-Thurston. So we can construct an approximation $\mathcal{A}'$ on S^2 from our K-approximation on Z by letting $p'(v)$ be the center of the circle associated to v , $r'(v)$ its radius, and U'_v the interiors of all triangles with a vertex at v. Having the same graph approximate both Z and S^2 gives us an obvious candidate for a quasimobius map between Z and S^2: for $x \in p(V) \subseteq Z$, let $f(x) = p'(p^{-1}(x))$. We will construct such functions on a sequence of graphs with mesh size tending to zero and apply the convergence result stated above. It remains to find conditions under which these maps are quasimobius.

The definition of quasimobius is symmetric in the sense that if f is quasimobius, then f^{-1} is also quasimobius. This is a consequence of the symmetry properties of the cross-ratio and the fact that the control function η is a homeomorphism with $\eta(t) \to 0$ as $t \to 0$. The following proposition says that, under favorable circumstances, we can relax the condition on the control function η.

Proposition 3. *Suppose $\mathcal{A}$ and $\mathcal{A}'$ are K-approximations of X and Y with the same underlying graph. Suppose $f : p(V) \to p'(V)$ is given by $f(x) = p'(p^{-1}(x))$. Suppose X and Y are LLC, doubling and homeomorphic to S^2, and suppose there is a function $\delta : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $\epsilon > 0$, we have $[x_1, x_2, x_3, x_4] < \epsilon$ whenever $[f(x_1), f(x_2), f(x_3), f(x_4)] < \delta(\epsilon)$. Then f is quasimobius.*

2.3 Modulus in graphs

In analogy to the definition of modulus in a metric measure space, we define modulus in a graph G: $mod_Q^G(A, B) = \inf \sum_{v \in V} w(v)^Q$, where the inf is taken over functions $w : V \to [0, \infty]$ such that $\sum_{v \in C} w(v) \geq 1$ for any chain C of vertices connecting the set A and the set B.

Proposition 4. *If $A' \subseteq N_K(A)$, $B' \subseteq N_K(A)$, then $mod_Q^G(A', B') \lesssim mod_Q^G(A, B)$.*

Proof. Given an admissible function w for (A, B), define $\tilde{w}(v) = \sum_{u \in B(v,K)} w(u)$. Then the assumption on the degree of G yields an upper bound on the number of vertices in $B(v, K)$, for any v, so $\sum_{v \in V} \tilde{w}(v)^Q \lesssim \sum_{v \in V} w(v)^Q$. It remains to show that w is admissible: if $u_1 \sim ... \sim u_n$ is a chain connecting A' to B', then $\cup_{i=1}^n B(u_i, K)$ contains a chain from A to B, so $\sum_{i=1}^n \tilde{w}(u_i) \geq 1$. □

For a set $E \subseteq Z$, let $V_E = \{v \in V : U_v \cap E \neq \emptyset\}$. The next few results relate the modulus of sets V_E, V_F in a graph to properties of the sets E and F in the metric space Z.

Proposition 5. *Let $Q \geq 1$, $\mathcal{A}$ a K-approximation of a Q-regular metric measure space Z, and let E, F be such that $dist(V_E, V_F)$ is not too small. Then $mod_Q(E, F) \lesssim mod_Q^G(V_E, V_F)$.*

The conditions on the separation of vertices (here and in what follows) should not cause alarm – we will consider graph approximations with very small mesh size, so most pairs of vertices will have large combinatorial distance in the graph. The assumptions are needed for technical reasons. To prove this proposition, we take an admissible function w for (V_E, V_F) and construct a function ρ admissible for (E, F) with a similar mass bound. Given a curve connecting E to F, we find the vertices that are close to the curve, then add up the values of the function w for those vertices, after scaling them appropriately to reflect the density of vertices near the curve. If we additionally assume that Z is Q-Loewner in the proposition above, i.e., if we assume Z is rectifiably connected and that there is a decreasing function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\phi(\Delta(E, F)) \leq mod_Q(E, F)$, then we have

$$\phi(\Delta(E, F)) \leq mod_Q^G(V_E, V_F) \tag{6}$$

if $dist(V_E, V_F)$ is not too small. Note that S^2 is 2-Loewner. The following proposition establishes the reverse inequality in certain situations.

Proposition 6. *Let $Q \geq 1$, and let $\mathcal{A}$ be a K-approximation of a Q-regular metric measure space Z. Then there exists $\psi : \mathbb{R}^+ \to (0, \infty]$ with $\psi(t) \to 0$ as $t \to 0$, such that*

$$mod_Q^G(V_E, V_F) \leq \psi(\Delta(E, F)). \tag{7}$$

We do not give any proof here since it is a bit technical, but it is very similar in spirit to the proof of the analogous fact for the traditional modulus in a Q-regular space. We define another notion of cross-ratio, this time for vertices in a graph. If $Q \geq 1$, define the *Ferrand cross-ratio* as

$[v_1, v_2, v_3, v_4]_Q = \inf mod_Q^G(A, B)$, where the inf is over all chains $A \ni v_1, v_3$ and $B \ni v_2, v_4$. The motivation for this definition is that in a Q-regular Q-Loewner space, $[x_1, x_2, x_3, x_4]$ is small iff there are continua $E \ni x_1, x_3$ and $F \ni x_2, x_4$ with small modulus, by Proposition 1.

Proposition 7. *If $Q \geq 1$, and Z is LLC with a K-approximation $\mathcal{A}$ satisfying* (7)*, then there is a $\delta_1 : \mathbb{R}^+ \to \mathbb{R}^+$ such that if $\epsilon > 0$, then $[v_1, v_2, v_3, v_4]_Q < \epsilon$ whenever $k(v_i, v_j) \geq 2(K + L)$ and $[p(v_1), p(v_2), p(v_3), p(v_4)] < \delta_1(\epsilon)$.*

Proof. If $[p(v_1), p(v_2), p(v_3), p(v_4)]$ is small, there are continua $E \ni p(v_1), p(v_3)$ and $E \ni p(v_2), p(v_4)$ with $\Delta(E, F)$ large, by Proposition 1. Cover E by neighborhoods U_{w_j}. By 3. of the definition of K-approximation, there is a chain $A \subseteq N_K(\cup w_j \subseteq N_K(V_E)$ connecting v_1 and v_3. Similarly, we find a chain B connecting v_2 and v_4. Now apply assumption 7 with Proposition 4 to get $mod_Q^G(A, B) \lesssim mod_Q^G(V_E, V_F) \lesssim \psi(\Delta(E, F))$. But $[v_1, v_2, v_3, v_4]_Q$ is the inf over all such chains A and B, and $\Delta(E, F)$ being large makes $\psi(\Delta(E, F))$ small, by Proposition 1, so we are done. ∎

Proposition 8. *If $Q \geq 1$, Z is LLC with a K-approximation $\mathcal{A}$ satisfying* (6)*, then there is a function $\delta_2 : \mathbb{R}^+ \to \mathbb{R}^+$ such that if $\epsilon > 0$, then $[p(v_1), p(v_2), p(v_3), p(v_4)] < \epsilon$ whenever $k(v_i, v_j) \geq$ is not too small and $[v_1, v_2, v_3, v_4]_Q < \delta_2(\epsilon)$.*

We skip the proof since it is similar to that of the previous proposition. Now we combine these propositions to obtain conditions guaranteeing the existence of a quasimobius map between graph approximations.

Proposition 9. *Let $Q \geq 1$. Let X be connected, with K-approximation $\mathcal{A} = (G, p, r, \mathcal{U})$ satisfying* (6)*. Let Y be LLC and doubling, with K-approximation $\mathcal{A}' = (G, p', r', \mathcal{U}')$ satisfying* (7)*. (Note that the underlying graphs are the same!) Let $W \subseteq V$ be a set of vertices that have large combinatorial separation (i.e., large enough to satisfy the separation conditions in* (6) *and* (7)*). Define $f : p(W) \to p'(W)$ in the obvious way, i.e., $f(x) = p'(p^{-1}(x)$. Then f is quasimobius, with control function depending quantitatively on the data.*

The reason we want the control function to depend only on the data is so that we can take a sequence of graphs with smaller and smaller mesh approximations, find a quasimobius map on each graph and then apply the convergence result stated above.

Proof. Our assumptions here allow us to apply Proposition 3.

All we need to show is that $[x_1, x_2, x_3, x_4]$ is small whenever $[f(x_1), f(x_2), f(x_3), f(x_4)]$ is small. But this follows by applying the previous two propositions in succession. □

As a corollary, we see that if $\mathcal{A}_k$ and $\mathcal{A}'_k$ are sequences of graph approximations with the same graphs G_k, and if mesh$(\mathcal{A}_k)$ and mesh$(\mathcal{A}'_k)$ go to zero, then we can apply the convergence result as long as the condition (5) is satisfied.

2.4 Proof of the main result

Proof. In Proposition 9, let $X = S^2$, and let Y be the 2-regular LLC space Z. By Proposition 2, there is a K-approximation of Z that is combinatorially equivalent to the 1-skeleton of a triangulation of S^2 with mesh$(\mathcal{A}_j) \leq \frac{1}{j}$. By the remarks following the proposition, we know that this gives us K'-approximations $\mathcal{A}'_j$ of Y, with the same underlying graphs $G_j = (V_j, E_j)$. Since S^2 satisfies (6), and since Z satisfies (7), we may apply Proposition 9 to find, for each j, a quasimobius map from $p(V_j) \to p'(V_j)$ with the same control function for all j (by the remarks following the proposition). Finally we remark that the condition (5) needed to apply the convergence result can be met by choosing three separated points $z_1, z_2, z_3 \in Z$, and using vertices $v_1^j, v_2^j v_3^j$ that approach z_1, z_2, z_3 , respectively, as $j \to \infty$. Then in the approximation of the sphere, we may actually choose to make $p(v_1^j), p(v_2^j)$ and $p(v_3^j)$ be well separated on the sphere. We can do this by applying a sequence of Mobius transformations to the sphere to send any three points to three equidistant points on a great circle. This allows us to apply the convergence lemma, so we are done. □

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3 Quasisymmetric rigidity of Sierpiński carpets

after M. Bonk and S. Merenkov [5]
A summary written by S. Zubin Gautam

Abstract

We summarize the results of [5], where it is shown that the group of quasisymmetric self-maps of the standard Sierpiński $\frac{1}{p}$-carpet S_p is finite; in particular, the group of quasisymmetric self-maps of S_3 is the dihedral group D_4. We also prove that any two distinct standard Sierpiński carpets are not quasisymmetrically equivalent.

3.1 Introduction

3.1.1 Carpets

For $p > 1$ odd, the *standard Sierpiński $\frac{1}{p}$-carpet* S_p is a self-similar subset of $\mathbb{R}^2$ obtained recursively by partitioning a square into p^2 subsquares of equal size, removing the open middle square, repeating this procedure on the $p^2 - 1$ remaining squares, and continuing indefinitely.[1] The construction is analogous to that of the usual Cantor sets in $\mathbb{R}$; in the following, we will view the carpets S_p as being constructed in the unit square $[0, 1] \times [0, 1]$ unless otherwise noted.

More generally, a *carpet* is a topological space homeomorphic to the standard Sierpiński carpet S_3 (or equivalently to any S_p). Carpets embedded in the sphere $\mathbb{S}^2$ can be characterized as sets $S = \mathbb{S}^2 \setminus \bigsqcup D_i$, where the D_i are Jordan domains with pairwise disjoint closures, such that S has empty interior and $\lim_{i\to\infty} \operatorname{diam}(D_i) = 0$ in the spherical metric.

In a general carpet S, a *peripheral circle* is a closed Jordan curve γ such that $S \setminus \gamma$ is connected. For carpets in the sphere as above, the peripheral circles are simply the boundaries ∂D_i.

In the sequel, we will glibly pass between the (extended) complex plane and the Riemann sphere $\mathbb{S}^2$ via the usual conformal stereographic projection. We call a carpet S a *carpet in K* if K is a closed Jordan domain, $S \subset K \subset \mathbb{S}^2$, and ∂K is a peripheral circle of S. S is a *square carpet in K* if its peripheral

[1] It will probably be useful to the reader to sketch pictures throughout this summary.

circles (possibly excepting ∂K) bound geometric squares with sides parallel to the real and imaginary axes in $\mathbb{C}$. Similarly, $S \subset \mathbb{S}^2$ is a *round carpet* if its peripheral circles are *bona fide* geometric circles.

We also consider carpets in cylinders $\mathbb{P}/T$, where $\mathbb{P}$ is the strip

$$\mathbb{P} = \{x + iy \in \mathbb{C} \mid 0 \leq y \leq 1\}$$

and T is a cyclic group of horizontal translations. These carpets simply arise as projections of T-invariant carpets in $\mathbb{P}$. The *bottom* and *top* peripheral circles are the projections of $\{y = 0\}$ and $\{y = 1\}$, respectively.

3.1.2 Quasisymmetric maps

A homeomorphism $f : X \to Y$ of metric spaces (X, d_X) and (Y, d_Y) is η - *quasisymmetric* if $\eta : [0, \infty) \to [0, \infty)$ is a homeomorphism and

$$\frac{d_Y\big(f(x), f(y)\big)}{d_Y\big(f(x), f(z)\big)} \leq \eta\left(\frac{d_X(x, y)}{d_X(x, z)}\right)$$

for all distinct $x, y, z \in X$. The homeomorphism η will usually be suppressed in the notation, and X and Y are said to be *quasisymmetrically equivalent* if there exists a quasisymmetric homeomorphism $f : X \to Y$. Heuristically, if y and z are symmetric about x, then f distorts the symmetry of the triple (x, y, z) by a bounded amount. Quasisymmetric maps serve as a natural generalization of quasiconformal maps to the setting of general metric spaces; in particular, the quasisymmetric self-maps of $\mathbb{R}^n$ are precisely the quasiconformal ones. See [8] for more on the basic theory of quasisymmetric maps.

Much of the motivation for studying quasisymmetric mapping properties of carpets arises from geometric group theory. To wit, any quasi-isometry of hyperbolic groups induces a quasisymmetric map between their boundaries at infinity, and carpets arise naturally as boundaries of Kleinian groups associated with hyperbolic 3-orbifolds. See [1] for a brief discussion of this connection.

3.1.3 Main results

The main results of [5] are the following theorems:

Theorem 1. *The group of quasisymmetric self-maps of the standard Sierpiński carpet S_3 is the dihedral group D_4 consisting of rotations and reflections of the square.*

Theorem 2. *The group of quasisymmetric self-maps of the standard Sierpiński carpet S_p is finite.*

Theorem 3. *The standard Sierpiński carpets S_p and S_q are not quasisymmetrically equivalent for $p \neq q$.*

3.2 Carpet modulus

The key tool used to prove the above theorems is a discrete analogue of the classical conformal modulus of a curve family in $\mathbb{S}^2$ (see [8]; the conformal modulus for curve families in $\mathbb{S}^2$ is the "2-modulus" in the notation therein).

Let S be a carpet in $\mathbb{S}^2$, and let ρ be a *mass distribution* (*i.e.*, a nonnegative function) defined on the set $\{C_i\}$ of its peripheral circles. The *ρ-length* of a curve $\gamma \subset \mathbb{S}^2$ is

$$l_\rho(\gamma) := \sum_{\gamma \cap C_i \neq \emptyset} \rho(C_i),$$

and the *total mass* of ρ is

$$\mathcal{M}(\rho) := \sum_i \rho(C_i)^2.$$

Given a curve family Γ in $\mathbb{S}^2$, a mass distribution ρ is Γ-*admissible* if there exists a subfamily $\Gamma_0 \subset \Gamma$ of conformal modulus zero such that $l_\rho(\gamma) \geq 1$ for all $\gamma \in \Gamma \setminus \Gamma_0$. The *modulus of* Γ *with respect to* S is

$$\text{mod}_S\Gamma := \inf_{\rho\ \Gamma\text{-admissible}} \mathcal{M}(\rho).$$

A mass distribution ρ is *extremal* for Γ if $\mathcal{M}(\rho) = \text{mod}_S\Gamma$; if an extremal ρ exists, it is unique by the usual ℓ^2 convexity argument. In the sequel, "modulus" will refer to this carpet modulus, while "conformal modulus" will refer to the classical 2-modulus.

The conformal modulus is a conformal invariant, but it is only *quasi*-invariant under quasiconformal maps. By contrast, it is easy to check that the carpet modulus just defined is genuinely invariant under quasiconformal self-maps of $\mathbb{S}^2$. The following monotonicity property of the modulus is crucial (and easily verified): Write $\Gamma \preceq \Gamma'$ if every curve $\gamma \in \Gamma$ has a subcurve in Γ'. Then $\Gamma \preceq \Gamma' \Rightarrow \text{mod}_S\Gamma \leq \text{mod}_S\Gamma'$.

3.3 Preliminary results

3.3.1 Group-like carpets

A closed Jordan curve $C \subset \mathbb{S}^2$ is a *quasicircle* if it is the image of a geometric circle under a quasi-Möbius map (the notion of a quasi-Möbius map is similar to that of quasisymmetry; for the sake of brevity we refer the reader to [9]). Curves in a family $\{C_i\}$ are *uniform quasicircles* if they are images of circles under quasi-Möbius maps of uniformly bounded distortion. The family $\{C_i\}$ is *uniformly separated* if there exists $\delta > 0$ such that

$$\frac{\operatorname{dist}(C_i, C_j)}{\min\{\operatorname{diam}(C_i), \operatorname{diam}(C_j)\}} \geq \delta$$

in the chordal metric on $\mathbb{S}^2$.

A carpet embedded quasisymmetrically in $\mathbb{S}^2$ whose peripheral circles are uniformly separated uniform quasicircles is said to be *group-like.* (The defining properties are enjoyed by all carpets that arise as boundaries of hyperbolic groups, whence the name.)

Lemma 4. *Any quasisymmetric image of a group-like carpet is again a group-like carpet. Moreover, any quasisymmetric map between group-like carpets extends to a quasiconformal self-map of* $\mathbb{S}^2$*.*

The first claim follows from basic properties of quasi-Möbius maps (see [3]); the second is proved via the classical Ahlfors-Beurling extension theorem (see [4] for details). In fact, the second claim holds even if one drops the uniform separation hypothesis on the peripheral circles.

By the invariance of the carpet modulus under quasiconformal homeomorphisms, we see that mod_S is a quasisymmetric invariant for *group-like* carpets S; *i.e.*, $\operatorname{mod}_S\Gamma = \operatorname{mod}_{f(S)} f(\Gamma)$ for all quasisymmetric f.

3.3.2 Assumed results and applications

We will assume the following three foundational uniformization and rigidity results:

Theorem 5 (Cylinder Uniformization Theorem)**.** *Let S be a group-like carpet with C_0 and C_1 two distinct peripheral circles of S. Then there exists a quasisymmetric map of S onto a square carpet in some cylinder $\mathbb{P}/T$ such that C_0 and C_1 are mapped to the bottom and top peripheral circles, respectively.*

Moreover, none of the peripheral circles in the square carpet is a point. (See [2].)

Theorem 6 (Round Uniformization Theorem)**.** *Every group-like carpet is quasisymmetrically equivalent to a round carpet in* $\mathbb{S}^2$*. (See [2]; see also [1] for a sketch of the proof.)*

The Round Uniformization Theorem is an analogue of Koebe's uniformization theorem for circle domains; in fact, Koebe's theorem is the basic ingredient of its proof.

Theorem 7 (Round Rigidity Theorem)**.** *If* S *is a round carpet in* $\mathbb{S}^2$ *of Hausdorff measure* 0*, then every quasisymmetric map of* S *onto another round carpet is the restriction of a Möbius transformation. (See [4].)*

We now apply these three theorems to establish some preliminary results.

Lemma 8 (Top-bottom lemma)**.** *Let* S *be a group-like carpet of measure zero, let* C_0 *and* C_1 *be two distinct peripheral circles, and let* Γ *be the curve family connecting* C_0 *and* C_1*. Then the extremal mass distribution* ρ_Γ *for* Γ *satisfies* $\rho_\Gamma(C_i) = \ell(C_i)$ *for* $i \notin \{0,1\}$*, where* $\ell(C_i)$ *is the side-length of the square in* $\mathbb{P}/T$ *corresponding to* C_i *under the Cylinder Uniformization Theorem.*

Proof sketch: After applying the Cylinder Uniformization map, consider the curve family Γ' of straight lines $\gamma_t := \{t\} \times [0,1]$ in $\mathbb{P}/T$ connecting the top and bottom peripheral circles. Say $[0,a) \times [0,1]$ is a fundamental domain for $\mathbb{P}/T$. Let ρ be any Γ'-admissible mass distribution; then

$$l_\rho(\gamma_t) = \sum_{\gamma_t \cap C_i \neq \emptyset} \rho(C_i) \geq 1, \;\; t \in [0,a) \setminus E,$$

where E is a measure-zero set accounting for the exceptional subfamily in the definition of admissibility. Integrating over $[0,a) \setminus E$ yields

$$a \leq \sum_i \ell(C_i)\, \rho(C_i) \leq \sqrt{a}\, \sqrt{\sum \rho(C_i)^2} = \sqrt{a\, \mathcal{M}(\rho)}$$

by Cauchy-Schwarz (recall that the peripheral squares C_i have sides parallel to the coordinate axes). So $\mathcal{M}(\rho) \geq a = \mathcal{M}(\rho_\Gamma)$ for all Γ'-admissible ρ, and hence ρ_Γ is extremal for Γ'. ρ_Γ is also Γ-admissible, so the claim follows from the monotonicity property of the modulus (since $\Gamma' \preceq \Gamma$).

Lemma 9. *Let S be a group-like carpet of measure 0 and C_0, C_1 two peripheral circles. Then the group of* orientation-preserving *quasisymmetric self-maps of S that fix C_0 and C_1 setwise is finite cyclic.*

Proof sektch: By Cylinder Uniformization, it suffices to assume S is a square carpet in a cylinder; suppose f is such a self-map. f has a quasiconformal extension $\tilde{f}$ by Lemma 4. By Lemma 8 and quasisymmetric invariance of the modulus, we see that f maps peripheral squares to peripheral squares of the same size. By considering vertical line segments connecting peripheral squares C to the top and bottom of the cylinder, one can also show that $f(C)$ has the same distance to the top and bottom as does C and that f maps vertices to vertices. Thus there is a rotation of the cylinder taking C to $f(C)$; using absolute continuity properties of f, one can in fact "glue" these rotations together, showing that f is itself a rotation of the cylinder. Since there are only finitely many peripheral squares of a given side-length, we conclude that the group in question is finite cyclic.

Theorem 10 (Three-Circle Theorem)**.** *Any two* orientation-preserving *quasisymmetric self-maps that act the same on a triple of distinct peripheral circles (or a triple of distinct points) must be the same map.*

This follows from the Round Uniformization and Round Rigidity Theorems; see [1] for details.

Lemma 11. *Let x and y be two distinct points on a single peripheral circle of a group-like carpet S. Then the group of quasisymmetric self-maps of S fixing x and y is cyclic.*

As above, this follows from Round Uniformization and Round Rigidity.

3.4 A preferred pair of peripheral circles

The "outer" and "middle" squares of a standard Sierpiński carpet S_p are defined in the obvious way.

Lemma 12. *Every quasisymmetric self-map of S_p preserves the outer and middle squares as a pair.*

Proof sketch: Let Γ be the curve family connecting the outer and middle squares, and let Γ_{ij} be that connecting the peripheral squares C_i and C_j, at least one of which is neither the middle nor the outer square. By the self-similarity of S_p, every curve in Γ_{ij} passes through a common "copy" of S_p (draw a picture); hence $\text{mod}_{S_p}(\Gamma_{ij}) \leq \text{mod}_{S_p}(\Gamma)$. If equality were to hold, then by uniqueness of extremal mass distributions, the extremal distribution for Γ_{ij} would have to concentrate on the aforementioned copy of S_p, which is a proper subset of S_p; this contradicts the "Top-bottom" lemma.

3.5 Proof of Theorem 2

We now prove that the group of quasisymmetric self-maps of S_p is finite. By Lemma 12, every element of this group preserves the outer and middle squares as a pair. By Lemma 9, the subgroup of orientation-preserving elements that fix these two squares is finite cyclic. The composition of any two orientation-reversing maps is orientation-preserving, and similarly the composition of two maps switching the outer and middle squares preserves them, so the whole group of quasisymmetric self-maps is in fact finite.

3.6 Weak tangents of S_p

A *weak tangent space* of a metric space X is a Gromov-Hausdorff limit of pointed dilations of X; see [7] and [6] for more useful and in-depth discussions. Heuristically, one "blows up" the infinitesimal structure of X near a chosen point. We isolate three particular weak tangents of S_p:

1. A $\frac{\pi}{2}$-*weak tangent* of S_p is any metric space isometric to

$$W_{\frac{\pi}{2}} := \bigcup_{n=0}^{\infty} p^n S_p$$

   equipped with the planar metric. The *vertex* of the $\frac{\pi}{2}$-weak tangent is the point corresponding to the origin. This is a weak tangent space to S_p at any corner point of the outer square; one can think of standing at the origin and "blowing up" the carpet S_p to fill up the entire first quadrant of the plane.

2. A π-*weak tangent* of S_p is any metric space isometric to

$$W_\pi := \bigcup_{n=0}^{\infty} p^n \big(S_p - \frac{1}{2}\big)$$

with the planar metric. As above, the *vertex* is the point corresponding to the origin in W_π, and a π-weak tangent is a weak tangent space to S_p at the midpoint of any side of any peripheral square.

3. A $\frac{3\pi}{2}$-*weak tangent* of S_p is any metric space isometric to $W_{\frac{3\pi}{2}}$, which is obtained by "gluing" three copies of $W_{\frac{\pi}{2}}$ together along the boundary axes; thus $W_{\frac{3\pi}{2}}$ may be taken to live in the first three quadrants of the plane. The *vertex* is again the point corresponding to the origin, and a $\frac{3\pi}{2}$-weak tangent is a weak tangent space to S_p at any corner point of any non-outer peripheral square.

These can all be seen to be group-like carpets. By applying Lemma 11 and completing each weak tangent by $\{\infty\}$ in the Riemann sphere, we see that the orientation-preserving quasisymmetric self-maps of any of these weak tangents fixing the vertex form a cyclic group. All of these groups contain all p^{th}-power multiplication maps in $W_{(\cdot)}$, so they are infinite cyclic.

Now let H be either the closed first quadrant, the closed upper half-plane, or the closed union of the first three quadrants. Let W be the corresponding weak tangent space of S_p in H as described above. If ψ is a quasisymmetric self-map of W fixing the vertex, Lemma 4 guarantees a quasiconformal extension $\tilde{\psi} : H \to H$.

Using Round Uniformization and Round Rigidity combined with Möbius transformations of the upper half-plane, one can show that the quotient space $H/\langle\tilde{\psi}\rangle$ is homeomorphic to a cylinder with boundary, with W projecting to a carpet $W/\langle\psi\rangle$ in this cylinder (strictly speaking one should omit the points 0 and ∞ from H). The boundary of W is mapped to the top and bottom peripheral circles. $W/\langle\psi\rangle$ is strictly a topological carpet, as there is no canonical metric on $H/\langle\tilde{\psi}\rangle$. One defines the modulus of a curve family similarly to the manner above, but in the definition of admissibility for mass distributions one takes an exceptional family $\Gamma_0 \subset \Gamma$ whose *preimage* under the quotient map $H \to H/\langle\tilde{\psi}\rangle$ has zero conformal modulus.

It is not hard to check that if $\psi = \phi^k$ for some $k \in \mathbb{Z}$ and Γ is the curve family connecting the top and bottom of such a cylinder, then $\mathrm{mod}_{W/\langle\psi\rangle}\Gamma = |\,k\,|\,\mathrm{mod}_{W/\langle\phi\rangle}\Gamma$.

Lemma 13. *There is no quasisymmetric map from* $W_{\frac{\pi}{2}}$ *to* $W_{\frac{3\pi}{2}}$ *sending the vertex to the vertex.*

Proof sketch: Let H_1 denote the first quadrant in the plane, and let H_2 denote the union of the first three quadrants. Suppose there is such a map

f. Let ϕ be a generator of the cyclic group G of quasisymmetric self-maps of $W_{\frac{\pi}{2}}$ fixing the vertex, and let $\tilde{\phi} : H_1 \to H_1$ be a quasiconformal extension of ϕ. Then the conjugate $\phi' = f\phi f^{-1}$ generates the group G' of quasisymmetric self-maps of $W_{\frac{3\pi}{2}}$ fixing the vertex. f admits a quasiconformal extension $\tilde{f} : H_1 \to H_2$, and $\tilde{\phi}' = \tilde{f}\tilde{\phi}\tilde{f}^{-1}$ is a quasiconformal extension of ϕ' to H_2.

$W_{\frac{\pi}{2}}/G$ and $W_{\frac{3\pi}{2}}/G'$ are carpets in the cylinders $H_1/\langle\tilde{\phi}\rangle$ and $H_2/\langle\tilde{\phi}'\rangle$ as described above; let Γ_G and $\Gamma_{G'}$ be the curve families connecting the tops and bottoms of the respective cylinders. Then it is not too hard to check that

$$\mathrm{mod}_{W_{\frac{3\pi}{2}}/G'}\Gamma_{G'} = \mathrm{mod}_{W_{\frac{\pi}{2}}/G}\Gamma_G.$$

Recall that $W_{\frac{3\pi}{2}}$ consists of three glued-together copies of $W_{\frac{\pi}{2}}$; let $\psi' \in G'$ agree with ϕ on one of these copies and extend by Schwarz reflection to the other copies. One checks that

$$0 < \mathrm{mod}_{W_{\frac{3\pi}{2}}/\langle\psi'\rangle}\Gamma_{\langle\psi'\rangle} \leq \frac{1}{3}\mathrm{mod}_{W_{\frac{\pi}{2}}/G}\Gamma_G = \frac{1}{3}\mathrm{mod}_{W_{\frac{3\pi}{2}}/G'}\Gamma_{G'}.$$

This is impossible; since ψ' is an element of the cyclic group G', $\mathrm{mod}_{W_{\frac{3\pi}{2}}/\langle\psi'\rangle}\Gamma_{\langle\psi'\rangle}$ must be an integer multiple of $\mathrm{mod}_{W_{\frac{3\pi}{2}}/G'}\Gamma_{G'}$ as remarked above.

3.7 Proof of Theorem 1

We now prove that the group of quasisymmetric self-maps of S_3 is the dihedral group D_4. View S_3 as obtained by subdividing $[0, 1] \times [0, 1]$ in the plane, and let f be a quasisymmetric self-map. By Lemma 12, f preserves the outer and middle squares as a pair; for now we assume f maps each of these squares to itself. Let C_0 be a peripheral square of size (side-length) $\frac{1}{9}$ such that the modulus of the curve family connecting C_0 to the outer square is maximal. Then using the self-similarity of S_3 and monotonicity of the modulus as in the proof of Lemma 12, one can see that C_0 must be mapped to another square of size $\frac{1}{9}$.

The eight squares of size $\frac{1}{9}$ are called "corner" or "middle" squares in the obvious manner. If C_0 and $f(C_0)$ are both corner squares, then there is a rotation or reflection of S_3 that acts the same as f on the outer square, the middle square, and C_0. The Three Circle Theorem then shows that $f \in D_4$. Now suppose C_0 is a corner square and $f(C_0)$ is a middle square. Without loss of generality we may assume C_0 is the "bottom left" square and $f(C_0)$ is the "bottom middle" square. If M is the reflection in the line $y = x$ and D is the reflection in the line $x = 1/2$, we see that f and MfD act the same on C_0 and the outer and middle squares. Again by the Three Circle Theorem, the two maps must coincide. Considering the fixed points of M and D, we see that $f(0,0) = (1/2,0)$ or $(1/2,1)$, and $f(1/3,1/3) = (1/2,1/3)$ or $(1/2,2/3)$. A quick examination of the induced maps on the weak tangents to S_3 at these points yields a contradiction to Lemma 13. If C_0 is a middle square, we can run the same argument backwards by considering f^{-1}.

To rule out the case where f interchanges the outer and middle squares, we refer the reader to the following proof of Theorem 3.

3.8 Proof of Theorem 3

To conclude, we prove that S_p and S_q are not quasisymmetrically equivalent for $p \neq q$. We will use the following lemma, whose proof follows along the lines of those of Lemma 8 and Lemma 9:

Lemma 14. *Let S_1 and S_2 be measure-zero square carpets in the unit square $K = [0,1] \times [0,1]$. If $f : S_1 \to S_2$ is an orientation-preserving quasisymmetric map that takes the vertices of K to the vertices and preserves the origin, then f is the identity map (so in particular $S_1 = S_2$).*

Now suppose $p \neq q$ and $f : S_p \to S_q$ is a quasisymmetric homeomorphism. By Lemma 12, f maps the outer square of S_p to either the outer or the middle square of S_q; suppose first that f maps the outer square to the outer square. Let $G_p \cong G_q$ denote the groups of quasisymmetric self-maps of S_p and S_q preserving the outer square, and let O_p and O_q denote the orbits of the origin under these respective groups. Note that the proof of Theorem 2 shows that $G_p \cong G_q \cong \mathbb{Z}_n \rtimes \mathbb{Z}_2$ for some $n = 4k$. Here $\mathbb{Z}_2$ can be taken to act by a reflection of the unit square, and there is a subgroup of $\mathbb{Z}_n \rtimes \mathbb{Z}_2$ isomorphic to D_4 acting by rotations and reflections of the square. Since f induces an isomorphism of G_p and G_q, we have that $f(O_p)$ is G_q-invariant; thus $f(O_p)$ is *a fortiori* D_4-invariant. If $f(0,0) \in O_q$, then post-composing with an element of G_q allows us to reduce to the case where $f(0,0) = (0,0)$ and f maps the vertices of the unit square to the vertices. (This is because f must conjugate a 90-degree rotation of the square to either itself or its inverse by the dihedral structure of $G_p \cong G_q$.) From here Lemma 14 yields a contradiction.

Now suppose $f(0,0) \notin O_q$, so $f(O_p) \cap O_q = \emptyset$. We claim that $(1/2,0) \notin O_p$; to see this, suppose that $\gamma \in G_p$ maps the origin to $(1/2,0)$. Then by exploiting the structure of G_p and applying the "three-point" version of the Three-Circle Theorem, we can show that $\gamma = M\gamma D$ as in the proof of Theorem 1. Just as in that proof, the ensuing discussion of weak tangents yields a contradiction. Now combining the fact that $(1/2, 0) \notin O_p$ with the fact that O_p contains the vertices of the unit square and is D_4-invariant, we obtain that $|O_p| = |f(O_p)| = 8n + 4$ for some $n \geq 0$. Exploiting the D_4-invariance of $f(O_p)$ and noting that $f(O_p)$ cannot contain the vertices of the unit square by assumption, we see that $(1/2, 0) \in f(O_p)$. But then we may as well assume $f(0, 0) = (1/2, 0)$, and an analogous "$f = MfD$" argument as above again yields a contradiction.

Similar arguments treat the case where f maps the outer square to the middle square; one replaces O_q with the G_q-orbit of the point $(\frac{q-1}{2q}, \frac{q-1}{2q})$. This completes the proof of Theorem 3.

Remark: The same argument actually rules out the case of quasisymmetric self-maps of S_p switching the outer and middle squares, so as a corollary of the proof we obtain the following refinement of Theorem 2:

Corollary 15. *For all p, the group of quasisymmetric self-maps of the standard Sierpiński carpet S_p is a semidirect product $\mathbb{Z}_n \rtimes \mathbb{Z}_2$ for some $n = 4k$.*

Actually, it is claimed in [1] that the group is dihedral, but no proof is provided. It is conjectured that in fact $S_p \cong D_4$ for *all* p; the $p = 3$ case is of course Theorem 1. It is claimed in [5] that similar methods provide the result for $p = 5$; the remaining cases are still open.

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4 Rigidity of Schottky sets

after M. Bonk, B. Kleiner and S. Merenkov [1]
A summary written by Hrant Hakobyan

Abstract

It is proven that every quasisymmetric homeomorphism of a Schottky set of spherical measure zero to another Schottky set is the restriction of a Möbius transformation of $\mathbb{S}^n$. In the other direction it is shown that every Schottky set in $\mathbb{S}^2$ of positive measure admits non-trivial quasisymmetric maps to other Schottky sets.

4.1 Introduction and main results

A subset S of the unit n-sphere $\mathbb{S}^n$ is said to be a *Schottky set* if its complement is a union of at least three disjoint open balls. The metric on S is the metric induced from $\mathbb{S}^n$.

A Schottky set $S \subset \mathbb{S}^n$ is said to be *rigid* if every quasi-symmetric map of S onto another Schottky subset of $\mathbb{S}^n$ extends to a Möbius transformation of $\mathbb{S}^n$.

Theorem 1. *Every Schottky set in $\mathbb{S}^n$, $n \geq 2$, of spherical measure zero is rigid.*

Theorem 2. *A schottky set in $\mathbb{S}^2$ is rigid if and only if it has spherical measure zero.*

Theorem 3. *For each $n \geq 3$ there is a Schottky set in $\mathbb{S}^n$ that has positive measure and is rigid*

One of the motivations for studying Schottky sets in $\mathbb{S}^n$ comes from the fact that every Schottky set can be thought of as "the boundary at infinity" of a convex region bounded by disjoint planes in the Hyperbolic space $\mathbb{H}^{n+1}$. So the results above can be translated to the corresponding results about quasi-isometric rigidity of such domains (meaning every quasi-isometry between such domains is actually a restriction of a hyperbolic isometry).

4.2 Connectivity properties of Schottky sets

First, any Schottky set can be written as

$$S = \mathbb{S}^n \setminus \bigcup_{i \in I} B_i, \tag{1}$$

where, $i \neq j$, implies $B_i \cap B_j = \emptyset$ and the index set I is either finite or is the set of natural numbers $\mathbb{N}$.

The collection of the $(n-1)$-spheres ∂B_i is the collection of the *peripheral spheres* of S. The proofs of the following results are quite elementary.

Lemma 4. *Let $S \subset \mathbb{S}^n$, $n \geq 2$, be a Schottky set, and B an open or (possibly degenerate) closed ball in $\mathbb{S}^n$. Then $S \cap B$ is path connected (or empty). In particular S is path connected.*

Proposition 5. *Let Σ be a topological $(n-1)$-sphere contained in a Schottky set $S \subset \mathbb{S}^n$, $n \geq 2$. Then $S \setminus \Sigma$ is connected if and only if Σ is a peripheral sphere of S.*

Corollary 6. *Every homeomorphism between Schottky sets S and S' in $\mathbb{S}^n$, $n \geq 2$, maps peripheral spheres of S to those of S'.*

4.3 Schottky groups

Let S be a Schottky set in $\mathbb{S}^n$ and let $R_i : \mathbb{S}^n \to \mathbb{S}^n$ be the reflection in the peripheral sphere ∂B_i. The subgroup of the group of Möbius transformations of $\mathbb{S}^n$ generated by the reflections R_i corresponding to S is denoted by Γ_S and will be called the *Schottky group associated to S* (strictly speaking Γ_S is a subgroup of the group generated by all reflections in spheres where Mobius transformations form an index 2 subgroup. It's easy to see that this doesn't affect the results in the paper). Note that for every element $U \in \Gamma_S$ there is a finite sequence $i_1, \ldots, i_k$ of indices from I, called a *reduced sequence* in the paper, such that $i_{k+1} \neq i_k$ and

$$U = R_{i_1} \circ \cdots \circ R_{i_k}.$$

Proposition 7. *The group Γ_S is discrete. Γ_S has a presentation given by the generators $R_i, i \in I$ and the only relations are $R_i^2 = id_{\mathbb{S}^n}, i \in I$.*

Proof. For the discreteness of Γ_S it is enough to find a $\delta > 0$ s.t.

$$\inf_{U \in \Gamma_S \setminus id} \left(\max_{x \in \mathbb{S}^n} |U(x) - x| \right) \geq \delta.$$

This is done by explicitly constructing δ.

The second part of the statement follows basically from the same argument as discreteness ☐

Next, the notion of Hausdorff convergence of subsets of a metric space is required. For that one first introduces the following notion of the *Hausdorff distance* between any two subsets A and B of a metric space (X, d_X)

$$dist_H(A, B) = \inf\{\delta > 0 : A \subset N_\delta(X) \text{ and } B \subset N_\delta(X)\},$$

where $N_\delta(A)$ denotes the open δ-neighborhood of A in X. Then a sequence $\{A_k\}$ of subsets of X is said to converge to $A \subset X$ if $dist_H(A_k, A) \to 0$. This is written as $A_k \to A$.

The following lemma is easy to see and is not proven in the paper.

Lemma 8. *If $B_k \to B \subset \mathbb{S}^n$ where B_k-s are closed balls and B is closed then B is a closed ball and $\partial B_k \to \partial B$. If $x \in int(B)$ then there is a $\delta > 0$ s.t. $B(x, \delta) \subset int(B_k)$ for large k.*

Let us denote

$$\mathcal{U} := \{U(B_i), U \in \Gamma_S, i \in I\}.$$

Lemma 9. *For every $\delta > 0$ only finitely many of the balls in $\mathcal{U}$ have diameter $\geq \delta$.*

Proof. The idea is to show that if the conclusion did not hold then Γ_S would be non-discrete, contradicting to Proposition 7. ☐

For the rest of the proof the following set would be very important to us

$$S_\infty := \bigcup_{U \in \Gamma_S} U(S).$$

Remark 10. *As a side remark one may notice that if S was a complement of finitely many disjoint balls then S_∞ would be the so called ordinary set of the Kleinian group Γ_S and the complement of S_∞, limit set $\Lambda(\Gamma_S)$, in this case would be just a Cantor set.*

Lemma 11. *For each point $x \in \mathbb{S}^n \setminus S_\infty$ there exists a unique sequence $U_k \in \mathcal{U}$ such that $U_{k+1} \subset U_k$ and $x \in \bigcap_{k\in\mathbb{N}_{>0}} U_k$.*

From the previous lemma it follows that $diam(U_k) \to 0$ and hence S_∞ is dense in $\mathbb{S}^n$.

Proof. This is done by using successive reflections in the boundary circles. □

4.4 Quasiconformal maps

In this section authors recall the definitions of the quasiconformal and quasi-Möbius maps and state the main extension theorems which are used later.

Theorem 12 (Ahlfors Beurling ($n = 2$) /Tukia Väisälä ($n \geq 3$))**.** *Let $n \geq 2$. Every H-qc map $f : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ has an H'-qc extension $F : \mathbb{R}^n \to \mathbb{R}^n$, where H' only depends on n and H.*

Proposition 13. *Let $D \neq \mathbb{S}^n$ and $D' \neq \mathbb{S}^n$ be closed non-degenerate balls in $\mathbb{S}^n, n \geq 2$ and $f : \partial D \to \partial D'$ a homeomorphism.*

(i) If f is η-quasi-Möbius, then it can be extended to an η'-quasi-Möbius map $F : D \to D'$, where η' only depends on n and η.

(ii) If each of the balls D and D' is contained in a hemisphere, and f is η-quasisymmetric, then f can be extended to an η' quasisymmetric map $F : D \to D'$, where η' only depends on n and η.

4.5 Extension of quasisymmetric maps between Schottky sets

This is one of the important sections of the paper. It is proved that every quasisymmetric map $f : S \to S'$ between two Schottky sets in $\mathbb{S}^n$ in fact extends to a global quasiconformal map of $\mathbb{S}^n$ which is equivariant with respect to the actions of Γ_S and Γ'_S. Given an element $U \in \Gamma_S$ with the presentation $U = R_{i_1} \circ \cdots R_{i_k}$ let $U' \in \Gamma_{S'}$ denote the element $U' := R'_{i_1} \circ \cdots R'_{i_k}$. First they note the following result, which is easy to prove.

Lemma 14. *There exists a unique bijection $f_\infty : S_\infty \to S'_\infty$ that extends f equivariantly, that is, $f_\infty|_S = f$ and $f_\infty \circ U = U' \circ f_\infty$, $\forall U \in \Gamma_S$.*

To obtain the main result of this section the extension theorems of section 4 are used to obtain a non-equivariant quasiconformal extension of f to $\mathbb{S}^n$.

Proposition 15. *Every quasysimmetric map between Schottky sets in* $\mathbb{S}^n, n \geq 2$ *extends to a quasiconformal homeomorphisn of* $\mathbb{S}^n$.

Next, one needs to modify the obtained qc map to obtain an equivariant qc map. This is done one step at a time as follows.

Suppose $T \subset \mathbb{S}^n, n \geq 2$, is a Schottky set, Σ a peripheral sphere of T, and R the reflection in Σ. Then $\tilde{T} = T \cup R(T)$ is also a Schottky set, called the *double* of T along Σ. Let T' be another Schottky set in $\mathbb{S}^n$, and $F : \mathbb{S}^n \to \mathbb{S}^n$ be an H-quasiconformal map with $F(T) = T'$. Then $\Sigma' = F(\Sigma)$ is a peripheral sphere of T'. Let R' be the reflection in Σ', and $\tilde{T}$ be the double of T' along Σ'. Denote by B the open ball with $\Sigma = \partial B$ and $B \cap T = \emptyset$. We define a map $\tilde{F} : \mathbb{S}^n \to \mathbb{S}^n$ by

$$\tilde{F}(x) = \begin{cases} F(x) & x \in \mathbb{S}^n \setminus B \\ R' \circ F \circ R(x) & x \in \bar{B}. \end{cases}$$

Lemma 16. *The map* $\tilde{F}$ *is an* H*-qc map with* $\tilde{F}|_T = F, \tilde{F}(\tilde{T}) = \tilde{T}'$*, and* $\tilde{F} \circ R = R' \circ \tilde{F}$.

So if there is an H-qc map of $\mathbb{S}^n$ mapping a Schottky set T to a Schottky set T', then there is a natural modification of this map to the corresponding doubles of T and T' which is also H-qc. Using the previous Lemma inductively one obtains the following result.

Lemma 17. $\exists H \geq 1$*, Schottky sets* S_k *and* S'_k *in* $\mathbb{S}^n$*, and* H*-qc maps* $F_k : \mathbb{S}^n \to \mathbb{S}^n$ *for* $k \in \mathbb{N}_0$ *with the following properties:*

(i) $F_0 = F, S_0 = S, S'_0 = S'$,

(ii) $F_k(S_k) = S_k, \forall k \in \mathbb{N}_0$,

(iii) $S_{k+1} \supset S_k$ *is a double of* S_k*, and* $S'_{k+1} \supset S'_k$ *is the corresponding double of* S'_k*, for all* $k \in \mathbb{N}_0$,

(iv) $F_k|_{S_k} = f_\infty|_{S_k}$ *for* $k \in \mathbb{N}_0$,

(v) $\cup_{k \in \mathbb{N}_0} S_k = S_\infty$.

Proposition 18. *The quasisymmeric map $f : S \to S'$ has an equivariant quasiconformal extension $F : \mathbb{S}^n \to \mathbb{S}^n$.*

Proof. Since the maps F_k of the previous Lemma are uniformly quasiconformal, they are also uniformly quasi-Möbius. Since $F_k|_{S_k} = f_\infty|_{S_k}$ it follows that f_∞ is a quasi-Möbius map from S_∞ to S'_∞. And since the latter sets are dense f_∞ extends to a quasi-Möbius map $F : \mathbb{S}^n \to \mathbb{S}^n$ which is then also equivariant since f_∞ is. □

4.6 Proof of theorem 1

Lemma 19. *Let $g : \mathbb{R}^n \to \mathbb{R}^n, n \in \mathbb{N}$ be a map that is differentiable at 0. Suppose there is a sequence $\{D_k\}$ of non-degenerate closed balls in $\mathbb{R}^n$ with diam$(D_k) \to 0$ s.t. $0 \in D_k$ and $D'_k = g(D_k)$ is a ball for all $k \in \mathbb{N}$.*

Then the derivative $Dg(0)$ of g at 0 is a (possibly degenerate or orientation reversing) conformal map, i.e. $Dg(0) = \lambda T$, where $\lambda \geq 0$ and $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear isometry.

Proof. Assume $g(0) = 0$. Let $\tilde{D} = \frac{1}{r_k} D_k$, where r_k is the radius of D_k. Then $\{\tilde{D}_k\}$ subconverges to a closed ball D of radius 1. Since $r_k \to 0$ the maps $g_k : \mathbb{R}^n \to \mathbb{R}^n$ defined as $g_k(x) = r_k^{-1} g(r_k x)$ converge to the linear map $L = Dg(0)$ locally uniformly on $\mathbb{R}^n$. Since $L(D)$ is a Hausdorff limit of the closed balls $r_k^{-1} D'_k = r_k^{-1} g(D_k) = g(\tilde{D}_k)$, it follows that D' is also a ball and hence L is conformal. □

Proof of Theorem 1. . By Proposition 18 there is an equivariant quasiconformal extension F of f to $\mathbb{S}^n$. We need to show that F is in fact conformal. For that all we need to show is that F is conformal almost everywhere, which is derived in the next paragraph using the previous Lemma 19. Indeed, from the analytic definition of quasiconformal maps it then follows that F is 1-quasiconformal and hence conformal.

Since $|S| = 0$ and S_∞ is a countable union of copies of S under Möbuis maps it follows that $|S_\infty| = 0$ and therefore $\mathbb{S}^n \setminus S_\infty$ has full measure. On the other hand every quasiconformal map F is differentiable, with invertible derivative almost everywhere. So a.e. point $x \in \mathbb{S}^n$ is a point in $\mathbb{S}^n \setminus S_\infty$ where F is differentiable. Therefore to finish the proof we need to show that at all such points the conditions of Lemma 19 are satisfied. But as shown before, every point in $\mathbb{S}^n \setminus S_\infty$ is contained in an infinite sequence of balls $\{D_k\}$ with diam$(D_k) \to 0$ s.t. every ball D_k is an image of a complementary ball of S

under some transformation from Γ_S. Since F maps peripheral spheres of S to peripheral spheres of S', and is equivariant, it follows that $F(D_k)$ is also a ball for every $k \in \mathbb{N}$. □

4.7 Proof of theorem 2

The main result of this section is a consequence of the Measurable Riemann Mapping Theorem (which is only available in dimension 2), which says that given any measurable function μ such that $\|\mu\|_\infty < 1$ there is a unique homeomorphic solution to the complex differential equation $w_{\bar{z}} = \mu w_z$, the Beltrami equation. The solution is in fact a quasiconformal map. Here uniqueness is understood in the sense that if there are two solutions they differ by a post-composition with a conformal map. So if one is looking for a solution of the Beltrami equation in $\overline{\mathbb{C}}$ with prescribed values at three points then the solution is unique. On the other hand for every qc map f of the plane there is a unique Beltrami coefficient $\mu_f \in L_\infty$ with $\|\mu\|_\infty < 1$.

One says that the Beltrami coefficient μ is invariant under a group of Möbius transformations Γ if

$$\mu(\gamma(z)) = \mu(z), \ \forall \gamma \in \Gamma \text{ and a.e. } z \in \mathbb{C}.$$

Lemma 20. *If Γ is a group of Möbius transformations and $F : \mathbb{C} \to \mathbb{C}$ is a qc map with a Γ invariant μ_F then $\Gamma' = F \circ \Gamma \circ F^{-1}$ is also a group of Möbius transformations.*

Lemma 21. *Let S be a Schottky set in $\mathbb{C}$ and $F : \mathbb{C} \to \mathbb{C}$ a qc map with a Γ_S invariant μ_F. Then $S' = F(S)$ is a Schottky set.*

Proof of Theorem 2. Suppose $|S| > 0$. Let ν be any nontrivial Beltrami coefficient on S, say $\nu = 1/2$ on S and $\nu = 0$ on the complement of S. Next define μ by spreading around ν in a Γ_S invariant fashion, i.e. let μ be defined on S_∞ as follows $\mu(\gamma(z)) = \nu(z)$ for every $\gamma \in \Gamma_S$ and every $z \in S$. Declare $\mu = 0$ on the complement of S_∞.

From the previous two lemmas it follows then that the qc solution F of the Beltrami equation with the Beltrami coefficient $\mu_F = \mu$ a.e. maps S onto a another Schottky set S' and since μ_F is $1/2 \neq 0$ on S_∞ which has positive measure if follows that it's not a Möbius transformation. □

4.8 Proof of theorem 3

The complement of at least three disjoint open balls in a domain $D \subset \mathbb{S}^n$ is called a *relative Schottky set in D*.

A relative Schottky set T in D is called *locally porous at* $x \in T$ if there is a neighborhood of x and constants $C \geq 1$ and $\rho > 0$ such that $\forall y \in t \cap U, \forall r \in (0, \rho)$ there is a subset B in the complement of T in D with $B(y, r) \cap B \neq \emptyset$ and $r/C \leq diam(B) \leq Cr$. If T is locally porous at every point then it's just *locally porous*. Note that a locally porous sets have zero measure (even though they may have Hausdorff dimension 2 unlike porous sets).

Theorem 3 follows from the following two facts (the proofs of which we omit):

Lemma 22. *Let $n \in \mathbb{N}, n \geq 3$, T and T' be relative Schottky sets in regions $D, D' \subset \mathbb{S}^n$, respectively, and $\psi : T \to T'$ a quasisymmetric map. If T is locally porous then ψ is the restriction of a Möbius transformation.*

Lemma 23. *Every region $D \subset \mathbb{S}^n$ contains a locally porous relative Schottky set.*

Proof of Theorem 3. Let D be a complement of a Cantor set $C \subset \mathbb{S}^n$ of positive measure and let $T \subset D$ be a locally porous relative Schottky set in D (given by the Lemma above). Then $T \cup C$ is a Schottky set in $\mathbb{S}^n$ of positive measure. Every quasisymmetric map f of $T \cup C$ onto another Schottky set in $\mathbb{S}^n$ restricts to a quasisymmetric map of T onto another relative Schottky set, which is a restriction of a Möbius transformation (by the first assumption) to D. Since D is dense in $\mathbb{S}^n$ it follows that f is the restriction of a Möbius transformation to $T \cup C$. □

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5 The Poincaré Inequality is an Open Ended Condition

after S. Keith and X. Zhong [4]
A summary written by Colin Hinde

Abstract

We prove a result of Keith and Zhong [4] stated succinctly in the title.

5.1 Introduction

Theorem 24. *Let (X, d, μ) and complete metric measure space with μ Borel and doubling, that admits a $(1, p)$-Poincaré inequality with $p > 1$. Then there exists $\epsilon > 0$ such that (X, d, μ) admits a $(1, q)$-Poincaré inequality for any $q > p - \epsilon$*

We begin by reviewing the necessary definitions to understand the statement of the theorem, and provide some additional context and consequences of this result. In section 2 we outline a proof of the main theorem. Throughout this chapter $C > 1$ is an abused constant; its value may vary from use to use, but will always depend only on the values of the constants associated to the hypotheses of the main theorem.

5.1.1 Definitions

(X, d, μ) throughout denotes a metric measure space with a regular Borel measure. For $E \subset X$ measurable $\mathrm{diam}(E)$ and $|E|$ denote the diameter and μ-measure of E. The metric ball $B(x, r) := \{y \in X : d(x, y) < r\}$ and $tB(x, r) = B(x, tr)$. μ is *doubling* if there exists and constant C such that $C|B(x, r)| \geq |2B(x, r)|$ for any ball. For a measurable function u we denote its mean value over on a set by $u_A := \frac{1}{|A|} \int_A u d\mu = ⨍_A u d\mu$.

Note that although the doubling condition is only defined in terms of a concentric ball with double the radius it implies more general volume comparison. For any ball $B(x, R)$, $y \in B(x, R)$ and $0 < r < R$

$$|B(x, R)| \leq C(R/r)^{\alpha} |B(y, r)|, \tag{2}$$

where the value of α depends on the doubling constant of X.

For a Lipschitz function $u : X \to \mathbb{R}$ we define

$$\text{Lip } u(x) = \limsup_{x \to y} \frac{|u(x) - u(y)|}{d(x, y).}$$

Now we are ready to define the Poincaré inequality.

Definition 25. *(X, d, μ) a metric measure space admits a $(1, p)$-Poincaré inequality, $p \geq 1$ with constants $C \geq 1, 0 < t < 1$ if every ball in X has finite, non-zero measure, and*

$$\fint_{tB} |u - u_{tB}| d\mu \leq C(\fint_{B} (Lip\ u)^p d\mu)^{1/p} \tag{3}$$

for any ball B and Lipschitz function u.

Notice that by Hölder's inequality a p-Poincaré inequality implies q-Poincaré for $q > p$. While for any $q < p$ one can construct a space with $(1 - p)$-Poincaré, but not $(1, q)$. According to [2] one way to achieve this is by gluing two copies of Euclidean $\mathbb{R}^n$ together along an appropriate Cantor set.

Of course Hölder's inequality does not help us improve the left hand side, so we close this section with the remark that restricting our attention to $(1-p)$-Poincaré, instead of considering (q, p) is not as restrictive as it at first appears. A result by Hasłasz and Koskela [1] demonstrates that for metric spaces with a doubling, Borel regular measure $(1, p)$-Poincaré inequality with $p \geq 1$ implies the (q, p) inequality with $q > 1$.

5.1.2 Remarks on Context and Consequences

We can think of metric spaces with a doubling measure, also called homogenous spaces, to be the most general class of spaces that admit a zero-order calculus. Then the addition of a Poincaré inequality will allow us to use many of the tools of first order calculus such as results from second order partial differential equations and Sobolev spaces, and the differentiability of Lipschitz functions. In this realm the small improvement seen in the main theorem here can prove vital.

Several problems from nonlinear potential theory can be solved given the hypothesis that one is working on $\mathbb{R}^n$ with a q-admissible measure (equivalently one that verifies the $(1-q)$-Poincaré inequality) where $q < p$, some critical dimension. For example in this scenario quasiminimizers of p-Dirichlet integrals satisfy Harnack's inequality, the strong maximum principle, and are Hölder continuous.

In another section of this conference we were introduced to various alternative definitions of Sobolev spaces. Shanmugalingam showed that Cheeger's Hajłasz's, and his own definitions for $(1, p)$-Sobolev space yield isometrically equivalent Banach spaces when we are in the realm of a homogenous metric space with a $(1 - q)$-Poincaré inequality, $q < p$. Again we may now improve the statement of this result to include the equality case.

We stress here that this open ended property is a property of the space itself, not just of the particular functions involved. We say that a pair of functions $u, g \in L^p(X)$ satisfy the $(1, p)$-Poincaré inequality if inequality (2) holds with g replacing Lip u in the right-hand side. Given Theorem 1, we could hope that such a pair would also satisfy a $(1, p - \epsilon)$ inequality regardless of whether or not the underlying space satisfies $(1, p)$-Poincaré. Unfortunately, this is not the case, Keith constructs an counter example on a Cantor space.

It has been noted by several authors that a Poincaré inequality, and hence a first order calculus, is in some way tied to the existence of a sufficient quantity of rectifiable curves in a space (e.g. [2], [5]). In an earlier paper, [3], Keith made this connection explicit by proving that a Poincaré inequality is equivalent to a bound on the modulus of the families of curves connecting separate points in the space. Although the proof here does not make use of this fact, perhaps it can aide one's intuition in understanding the nature of the open-endedness of the condition by considering the modulus inequality as well.

5.2 Outline of Proof of Theorem 1.

We begin by making simplifying assumption that (X, d, μ) is in fact a geodesic metric space. This can be done without loss of generality because any homogenous metric space that admits a Poincaré inequality is bi-Lipschitz equivalent to a geodesic space. This assumption is simplifying, beyond ensuring the existence of geodesics, because in a geodesic metric space the parameter t in the Poincaré inequality can be taken to be 1 (by perhaps increasing the constant C). Also we can now control the measure of a thin outer shell of a ball by

$$|B(x, r) \backslash B(x, r - \delta r)| \leq \delta^\alpha |B(x, r)|, \tag{4}$$

where α depends only on the doubling constant.

The proof of the main theorem will require the use of (some variants of) the sharp fractional maximal operator. For Lipschitz u

$$M^\sharp u(x) = \sup_B \frac{1}{\operatorname{diam}(B)} \fint |u - u_B| d\mu, \tag{5}$$

where the supremum is taken over all balls that contain x.

It is a straightforward but useful fact that we can control Lip u on sets where $M^\sharp u$ is bounded. For the variants we shall encounter later, statements close to the following hold.

Proposition 26. *(X, d, μ) a metric measure space with μ doubling and u a Lipschitz function. Then there exists $C > 0$ depending only on the doubling constant such that for all $r > 0, y \in X$ and $x \in B(y, r)$*

$$|u(x) - u_{B(y,r)}| \le C r M^\sharp u(x) \tag{6}$$

which implies that the restriction of u to the set $x \in X : M^\sharp u(x) \le \lambda$ is $2C\lambda$-Lipschitz.

The proof of the main theorem is quite long and technical. The presentation here is backwards from the original with hopes that the reader will better appreciate the implications of some of the preliminary technical estimates by the time they are mentioned.

The proof is completed by integrating an estimate of the measure of the super-level sets of a constrained version of the sharp fractional maximal operator. Fix a ball $\tilde{B} \subset X$. For $t \ge 1$ we have

$$M_t^{\sharp *} u(x) = \sup_B \frac{1}{\operatorname{diam} B} \fint_B |u - u_B| d\mu, \tag{7}$$

for any Lipschitz u and $x \in \tilde{B}$; the supremum is taken over all balls such that $tB \subset \tilde{B}$ and $x \in B$. Notice that this definition yields

$$M_{40}^{\sharp *} u(x) \ge \frac{1}{\operatorname{diam} B'} \fint_{B'} |u - u_{B'}| d\mu, \tag{8}$$

where 40 is some rather arbitrary large number, and $B' = \frac{1}{40}\tilde{B}$.

Let $U_\lambda^* = \{x \in \tilde{B} : M_{40}^{\sharp *} u(x) > \lambda\}$. We can show using local estimates that for any $\alpha \in \mathbb{N}$ there exists $k_2 \in \mathbb{N}$ such that for all integer $k \geq k_2$ and every $\lambda > 0$ we have

$$|U_\lambda^*| \leq 2^{kp-\alpha}|U_{2^k\lambda}^*| + 8^{kp-\alpha}|U_{8^k\lambda}^*| + 10^{kp}|\{x \in \tilde{B} : \text{Lip } u(x) > 10^{-k}\lambda\}|. \quad (9)$$

Next we integrate both sides of the $\alpha = 3$ version of this inequality from 0 to ∞ against the measure $d\lambda^{p-\epsilon}$, where $0 < \epsilon < p-1$ is chosen so that $8^{ke} < 2$. The proof is then essentially complete; one only needs to translate the integration with respect to the parameter λ into integration on X and apply the observation of inequality (7) to finish. In the final subsection we will sketch the techniques used to arrive at the local estimates and, thus, estimate (8).

5.2.1 Technical Estimates

The local estimate is based on another small variant of the sharp fractional maximal operator. Fix a ball $X_1 \subset X$ and let $X_i = 2^{i-1}X_1$. Given a Lipschitz function u, let

$$M_i^\sharp u(x) = \sup_B \frac{1}{\text{diam } B} \fint_B |u - u_B| d\mu, \quad (10)$$

be defined for any $x \in X_{i+1}$, with the supremum taken over all balls $B \subset X_{i+1}$ that contain x. We will let $U_\lambda := \{x \in X_4 : M_4^\sharp u(x) > \lambda\}$ denote the super-level sets. The estimate is as follows

Proposition 27. *Let $\alpha \in \mathbb{N}$. There exists $k_1 \in \mathbb{N}$ that depends only on C and α such that for all integer $k \geq k_1$ and every $\lambda > 0$ with*

$$\frac{1}{\textit{diam } X_1} \fint_{X_1} |u - u_{X_1}| d\mu > \lambda, \quad (11)$$

we have

$$|X_1| \leq 2^{kp-\alpha}|U_{2^k\lambda}| + 8^{kp-\alpha}|U_8^k\lambda| + 8^{k(p+1)}|\{x \in X_5 : \textit{Lip } u(x) > 8^{-k}\lambda\}|. \quad (12)$$

We can simplify the exposition of the proof by rescaling u, d, and μ to assume without loss of generality that $\lambda = \text{diam} X_1 = |X_1| = 1$. The proof then is by contradiction - we assume that the proposition does not hold and that therefore for large enough values of k $|U_{2^k}| < 2^{-kp+\alpha}$, $|U_{8^k}| < 8^{-kp+\alpha}$, and $|\{x \in X_5 : \text{Lip} u(x) > 8^{-k}\}| < 8^{-k(p+1)}$. What follows is a sequence of lemmas to find lower bounds for such a k eventually leading to a contradiction.

Lemma 28. *We have*

$$\int_{X_2\setminus U_{2^k}} |u - u_{X_2\setminus U_{2^k}}|d\mu \geq 1/C$$

Loosely speaking, u has some oscillation outside of U_{2^k}. The proof uses little more than assumed contradiction, the above mentioned properties of geodesic doubling spaces, and the cleverly defined set of balls that overlap with U_{2^k} on between 1/4 and 3/4 of their total volume.

Next we can preserve these large scale oscillations while smoothing on the small scale by taking advantage of the Lipschitz property of sub-level sets of maximal type operators.

Lemma 29. *There exists a $C8^k$-Lipschitz extension f of $u|_{X_3\setminus U_{8^k}}$ to X_3 such that*

$$M_2^\sharp f(x) \leq CM_4^\sharp u(x)$$

for every $x \in X_2 \setminus U_{8^k}$

Now, finally getting to use the fact that we are in a $(1-p)$-Poincaré space, we find that that the p-norm of Lip$f$ is small outside of the superlevel sets of u. Meanwhile we define another new function, h, via Lipschitz extensions of f, and demonstrate a lower bound for the p-norm of Lip$h$. These Poincaré type estimates lead to a contradiction (in which it is crucial that $p > 1$), thus proving Proposition 4.

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6 Distortion of Hausdorff measures and removability for quasiregular mappings

after K. Astala, A. Clop, J. Mateu, J. Orobitg and I. Uriarte-Tuero [1]
A summary written by Vjekoslav Kovač

Abstract

We study absolute continuity properties of pull-backs of Hausdorff measures under K-quasiconformal mappings, especially at the critical dimension $\frac{2}{K+1}$. We also consider the analogue of the Painlevé problem for bounded and BMO K-quasiregular mappings.

6.1 Notation and basic definitions

Let us first introduce all concepts that appear in the text. The main purpose of the paper [1] is to prove several new results relating those notions and compare them to previously known results.

Throughout this text $\mathcal{H}^\alpha$ denotes the α-dimensional *Hausdorff measure* and $\mathcal{M}^\alpha$ denotes the α-dimensional *Hausdorff content*.

More generally, for every continuous non-decreasing function $h\colon [0,\infty) \to [0,\infty)$ such that $h(0) = 0$ (any such function is called a *measure function*) and for every $F \subset \mathbb{C}$, we can define

$$\mathcal{M}^h(F) = \inf \textstyle\sum_j h(\delta_j),$$

where the infimum is taken over all countable coverings of F by disks with diameters δ_j. Then $\mathcal{M}^\alpha$ becomes a special case for the choice $h(t) = t^\alpha$. We also denote by $\mathcal{F}_\alpha$ the class of all measure functions such that $h(t) \leq t^\alpha$ and $\lim_{t\to 0} \frac{h(t)}{t^\alpha} = 0$. Finally, the *lower* α-dimensional *Hausdorff content* of F is defined to be

$$\mathcal{M}^\alpha_*(F) = \sup_{h\in\mathcal{F}_\alpha} \mathcal{M}^h(F).$$

We say that f is a K*-quasiregular mapping* in a domain $\Omega \subset \mathbb{C}$ if it belongs to the Sobolev space $W^{1,2}_{loc}(\Omega)$ and satisfies the *distortion inequality*:

$$\max_{\alpha\in\mathbb{C},\,|\alpha|=1} |\partial_\alpha f| \ \leq\ K \min_{\alpha\in\mathbb{C},\,|\alpha|=1} |\partial_\alpha f| \qquad \text{a.e. in } \Omega.$$

Here $\partial_\alpha f$ denotes the directional derivative of f, taken in the sense of distributions.

A K-*quasiconformal mapping* is a homeomorphism $\varphi\colon \Omega \to \Omega'$ between two planar domains $\Omega, \Omega' \subset \mathbb{C}$ that is also K-quasiregular. In order to normalize quasiconformal mappings $\varphi\colon \mathbb{C} \to \mathbb{C}$, we say that φ is a *principal* K-*quasiconformal mapping* if it is conformal outside a compact set and satisfies $\varphi(z) - z = O(\frac{1}{|z|})$ as $|z| \to \infty$.

Now we introduce several notions of removability. A compact set E is said to be *removable for bounded (resp.* BMO*) analytic functions* if for any open neighborhood Ω of E, every bounded (resp. BMO($\mathbb{C}$)) analytic function on $\Omega \setminus E$ (resp. $\mathbb{C} \setminus E$) has an analytic extension to Ω.

We say that a set E is *removable for bounded (resp.* BMO*)* K*-quasiregular mappings* if every K-quasiregular mapping in $\mathbb{C} \setminus E$ which is in $L^\infty(\mathbb{C})$ (resp. BMO($\mathbb{C}$)) admits a K-quasiregular extension to $\mathbb{C}$.

A quantitive notion of removability is called *capacity* — again we have several versions of it. Remember that for a function f holomorphic in a neighborhood of $\infty \in \overline{\mathbb{C}}$ we define $f'(\infty) = \lim_{|z|\to\infty} z\,(f(z) - f(\infty))$.

Let $E \subset \mathbb{C}$ be a compact set. The *analytic capacity* of E is

$$\gamma(E) = \sup\{|f'(\infty)| : f \in \mathrm{H}(\overline{\mathbb{C}} \setminus E) \cap \mathrm{L}^\infty(\mathbb{C} \setminus E),\ f(\infty) = 0,\ \|f\|_\infty \le 1\}.$$

The *BMO analytic capacity* of E and the *VMO analytic capacity* of E are respectively defined to be:

$$\gamma_0(E) = \sup\{|f'(\infty)| : f \in \mathrm{H}(\overline{\mathbb{C}} \setminus E) \cap \mathrm{BMO}(\mathbb{C}),\ f(\infty) = 0,\ \|f\|_{\mathrm{BMO}} \le 1\},$$

$$\gamma_*(E) = \sup\{|f'(\infty)| : f \in \mathrm{H}(\overline{\mathbb{C}} \setminus E) \cap \mathrm{VMO}(\mathbb{C}),\ f(\infty) = 0,\ \|f\|_{\mathrm{BMO}} \le 1\}.$$

For any pair $\alpha > 0$, $p > 1$ with $0 < \alpha p < 2$, one defines the *Riesz capacity* of E by

$$\dot{C}_{\alpha,p}(E) = \sup_\mu \mu(E)^p,$$

where the supremum runs over all positive measures μ supported on E and such that $\|I_\alpha * \mu\|_{p'} \le 1$. Here $I_\alpha(z) = \frac{1}{|z|^{2-\alpha}}$ and $\frac{1}{p} + \frac{1}{p'} = 1$.

6.2 Quasiconformal distortion of Hausdorff measures

It is well known that quasiconformal mappings preserve sets of zero Hausdorff dimension. However, they do not preserve Hausdorff dimension in general. A result by Astala states that for every compact set E with dimension t and for every K-quasiconformal mapping φ we have

$$\frac{1}{K}\left(\frac{1}{t} - \frac{1}{2}\right) \le \frac{1}{\dim(\varphi(E))} - \frac{1}{2} \le K\left(\frac{1}{t} - \frac{1}{2}\right).$$

Moreover, both equalities may occur, so the above bounds are optimal.

Some of the questions we study here are analogous estimates at the level of Hausdorff measures $\mathcal{H}^t$. This leads us to the following conjecture:

Conjecture 1. *Let $E \subset \mathbb{C}$ be a compact set and let $\varphi\colon \mathbb{C} \to \mathbb{C}$ be a K-quasiconformal mapping.*

(a) For any $t \in [0,2]$ denote $t' = \frac{2Kt}{2+(K-1)t}$. We ask if it is true that

$$\mathcal{H}^t(E) = 0 \;\Rightarrow\; \mathcal{H}^{t'}(\varphi(E)) = 0.$$

(b) If φ is also conformal on $\mathbb{C} \setminus E$, is it true that for $t \in (0,2]$ we have

$$\mathcal{M}^t(\varphi(E)) \sim \mathcal{M}^t(E)$$

with constants depending only on K and t?

Let us remark that classical results of Ahlfors and Mori assert that the statement (a) is true in extremal cases $t = 0$ and $t = 2$. As the main result of this section we prove that statement for $t = \frac{2}{K+1}$, $t' = 1$.

Theorem 2. *Let E be a compact set and let $\varphi\colon \mathbb{C} \to \mathbb{C}$ be K-quasiconformal.*

(a) $\mathcal{M}^1(\varphi(E)) \le C \left(\mathcal{M}^{\frac{2}{K+1}}(E)\right)^{\frac{K+1}{2K}}$

(b) $\mathcal{H}^{\frac{2}{K+1}}(E) = 0 \;\Rightarrow\; \mathcal{H}^1(\varphi(E)) = 0$

(c) If φ is also principal and conformal on $\mathbb{C} \setminus E$, then

$$\mathcal{M}^1(\varphi(E)) \sim \mathcal{M}^1(E).$$

Remark. Part (b) is what we call *absolute continuity of Hausdorff measures $\mathcal{H}^\alpha$*. Observe that it is an immediate consequence of part (a), since $\mathcal{M}^\alpha$ and $\mathcal{H}^\alpha$ have the same zero sets.

Sketch of the proof. We first prove part (c) by restating it in terms of the BMO analytic capacity. Because of Verdera's result $\mathcal{M}^1(E) \sim \gamma_0(E)$ it suffices to show $\gamma_0(\varphi(E)) \sim \gamma_0(E)$, but this follows immediately from the definition of γ_0.

To prove part (a) we may WLOG assume that E is a subset of the unit disk $\mathbb{D}$, and that φ is principal. Then we decompose $\varphi = h \circ \varphi_1$, where h and

φ_1 are principal K-quasiconformal maps, φ_1 is conformal in $\Omega\cup(\mathbb{C}\setminus\mathbb{D})$ and h is conformal outside $\varphi_1(\overline{\Omega})$, for some appropriately small open neighborhood Ω of the set E. Finally we use part (c) applied to h to obtain $\mathcal{M}^1((h\circ\varphi_1)(E)) \leq C\mathcal{M}^1(\varphi_1(\Omega))$ and then some rough area estimates to control $\mathcal{M}^1(\varphi_1(\Omega))$. ∎

Another interesting result in the same direction is:

Theorem 3. *Let $E \subset \mathbb{C}$ be a compact set and let $\varphi\colon \mathbb{C} \to \mathbb{C}$ be a K-quasiconformal map.*

(a) If $\mathcal{H}^{\frac{2}{K+1}}(E)$ is σ-finite, then $\mathcal{H}^1(\varphi(E))$ is also σ-finite.

(b) If $\mathcal{M}_*^{\frac{2}{K+1}}(E) = 0$, then $\mathcal{M}_*^1(\varphi(E)) = 0$.

Sketch of the proof. From the result of Sion and Sjerve we know that $\mathcal{M}_*^\alpha(F) = 0$ if and only if F is a countable union of sets with finite α-dimensional Hausdorff measure. Therefore parts (a) and (b) are equivalent. We may also assume that φ is principal.

Now we pass to the VMO capacity using another Verdera's result, $\mathcal{M}_*^1(E) \sim \gamma_*(E)$ and observe that $\gamma_*(\varphi(E)) \sim \gamma_*(E)$. The rest of the proof is similar to the proof of the previous theorem. ∎

The last result of this type is the quasiconformal invariance of the Riesz capacities.

Theorem 4. *Let $\varphi\colon \mathbb{C} \to \mathbb{C}$ be a principal K-quasiconformal mapping, which is also conformal on $\mathbb{C} \setminus E$. Let $1 < p < 2$ and $\alpha = \frac{2}{p} - 1$. Then*

$$\dot{C}_{\alpha,p}(\varphi(E)) \sim \dot{C}_{\alpha,p}(E),$$

with constants depending only on K and p.

Sketch of the proof. The main idea is to use the following result about Riesz capacities:

$$\dot{C}_{\alpha,p}(E)^{1/p} \sim \sup\{|f'(\infty)| : f \in \mathrm{H}(\mathbb{C} \setminus E),\ f(\infty) = 0,\ \|f\|_{\dot{W}^{1-\alpha,p'}} \leq 1\}.$$

Here $\dot{W}^{1-\alpha,p'}(\mathbb{C})$ denotes the homogenous Sobolev space. Therefore it suffices to show that $\gamma_{1-\alpha,p'}(\varphi(E)) \leq C_K\gamma_{1-\alpha,p'}(E)$ and the latter follows from the fact that every K-quasiconformal map φ induces a bounded linear operator on $\dot{W}^{1-\alpha,p'}(\mathbb{C})$ given by $f \mapsto f \circ \varphi$. ∎

6.3 Quasiconformal distortion of 1-rectifiable sets

Recall that a set $E \subset \mathbb{C}$ is 1-*rectifiable* if there exists a set E_0 of zero length such that $E \setminus E_0$ is contained in a countable union of Lipschitz curves. In this section we basically show that (up to sets of zero length) a K-quasiconformal image of any 1-rectifiable set has dimension strictly larger than $\frac{2}{K+1}$.

Theorem 5. *Suppose that E is a 1-rectifiable set, and let $\varphi : \mathbb{C} \to \mathbb{C}$ be a K-quasiconformal mapping. Then there exists a subset $E_0 \subset E$ of zero length such that* $\dim(\varphi(E \setminus E_0)) > \frac{2}{K+1}$.

Sketch of the proof. We may assume that E is a subset of the unit circle $\partial\mathbb{D}$, with $\dim(E) = 1$. Then we recall (from a result by Lehto and Virtanen) that whenever φ is K-quasiconformal and $K = K_1 K_2 \dots K_n$ for some $K_j > 1$, then φ can be factored as $\varphi = \varphi_n \circ \dots \circ \varphi_2 \circ \varphi_1$, where each φ_j is K_j-quasiconformal. This allows us to assume that K from the statement of the theorem can be as close to 1 as we wish. Finally, the theorem follows from this technical lemma:

Lemma 6. *There exist constants c_0, c_1, $c_2 > 0$ such that if $E \subset \partial\mathbb{D}$, $0 < \varepsilon < c_0$ and $f \colon \mathbb{C} \to \mathbb{C}$ is $(1+\varepsilon)$-quasiconformal, then*

$$\dim(E) \geq 1 - c_1\,\varepsilon^2 \;\Rightarrow\; \dim(\varphi(E)) \geq 1 - c_2\,\varepsilon^2.$$

6.4 A removability theorem for quasiregular mappings

The classical *Painlevé problem* consists of giving metric and geometric characterizations of those sets E that are removable for bounded analytic functions, or equivalently $\gamma(E) = 0$. The Painlevé theorem states that $\mathcal{H}^1(E) = 0$ implies $\gamma(E) = 0$. In particular, all sets E with Hausdorff dimension strictly smaller than 1 have $\gamma(E) = 0$, while Ahlfors showed that sets E with dimension strictly larger than 1 satisfy $\gamma(E) > 0$. However, the exact characterization of sets with zero analytic capacity is more complicated. David and Tolsa proved that among the 1-dimensional sets with σ-finite length we have $\gamma(E) = 0$ if and only if E is purely unrectifiable, i.e. $\mathcal{H}^1(E \cap \Gamma)$ for all rectifiable curves Γ.

If we consider the removability for BMO analytic functions, the characterising property is simply $\mathcal{H}^1(E) = 0$, as is proved by Kaufman.

Here we consider removability for bounded/BMO K-quasiregular mappings, and now the dimension $\frac{2}{K+1}$ turns out to be critical.

Theorem 7. *Let E be a compact set in the plane and let $K > 1$.*

(a) If $\mathcal{H}^{\frac{2}{K+1}}(E) = 0$, then E is removable for BMO *K-quasiregular maps.*

(b) If $\mathcal{H}^{\frac{2}{K+1}}(E)$ is σ-finite, then E is removable for bounded K-quasiregular mappings.

(c) For K-quasiconformal map φ the image $\varphi(E)$ is purely unrectifiable.

Sketch of the proof. For the proof of part (a), take $f \in \mathrm{BMO}(\mathbb{C})$ which is also K-quasiregular on $\mathbb{C} \setminus E$. We can decompose it as $f = F \circ \varphi$, where φ is K-quasiconformal and $F \in \mathrm{BMO}(\mathbb{C})$ is holomorphic on $\mathbb{C} \setminus \varphi(E)$. From the theorem 2 we know that $\mathcal{H}^1(\varphi(E)) = 0$. Thus, by Kaufman's result $\varphi(E)$ is removable for BMO analytic functions. In particular F extends to an entire function and f extends to a K-quasiregular map on the whole $\mathbb{C}$.

For the part (b) we use the same decomposition $f = F \circ \varphi$ and then apply theorem 3 to the map φ. Since $\varphi(E)$ has σ-finite length, from Besicovitch's result it can be decomposed as $\varphi(E) = \bigcup_n (R_n \cup U_n \cup B_n)$, where R_n are 1-rectifiable sets, U_n are purely 1-unrectifiable sets and sets B_n have zero length. Finally we use countable semiadditivity of analytic capacity, the result by David and Tolsa which implies $\gamma(U_n) = 0$ and theorem 5 which implies $\gamma(R_n) = 0$. Therefore $\gamma(\varphi(E)) = 0$ and we are done.

Part (c) is an immediate consequence of (b). ■

We end this text with the following conjecture, which would imply that the result in part (a) above is sharp.

Conjecture 8. *For every $K \geq 1$ there exists a compact set E with $0 < \mathcal{H}^{\frac{2}{K+1}}(E) < \infty$, such that E is not removable for some K-quasiregular mapping in* $\mathrm{BMO}(\mathbb{C})$.

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7 The quasiconformal Jacobian problem

after M. Bonk, J. Heinonen, and E. Saksman [1]
A summary written by Peter Luthy

Abstract

Quasiconformal maps have been a fundamental part of complex analysis since the 1930s. Their applications stretch far beyond strictly analytic problems. Despite the longevity of the idea, it is unknown (in an analytic sense) which functions are comparable to the Jacobian of such a map. We begin by exploring an interesting relationship between metric doubling measures and bi-Lipschitz maps on $\mathbb{R}^2$. From this, we will produce a rather broad class of functions comparable to the Jacobian of a quasiconformal map.

7.1 Introduction

As stated in the abstract, quasiconformal mappings of the plane have been in the mathematical vernacular for quite some time. That they survived into the modern era is due in no small part to their critical role in proving big theorems. Their creation, however, resulted from a rather small problem. The Riemann mapping theorem guarantees the existence of a holomorphic map from a square to a rectangle (not a square). However, this map does not carry vertices to vertices. In fact, no such holomorphic map exists. Grötzsch asked whether there was a best "almost holomorphic" map which had this desired property. In doing so, he produced an early definition of quasiconformal maps.

The quasiconformal Jacobian problem asks the following question: which functions w are a.e. comparable to the Jacobian of a quasiconformal map f? By this we mean to determine for which functions w there is a quasiconformal map $f : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying the estimate

$$\frac{1}{C}w(x) \leq J_f(x) \leq Cw(x) \text{ for a.e. } x \in \mathbb{R}^2$$

for some $C > 0$. To proceed with our discussion we shall need some terminology:

Definition 1. *The* local 1,2 Sobolev space $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2)$ *is the class of functions in* $L^2_{\mathrm{loc}}(\mathbb{R}^2)$ *whose first-order weak derivatives are in* $L^2_{\mathrm{loc}}(\mathbb{R}^2)$*.*

Definition 2. *A homeomorphism* $f : \mathbb{R}^2 \to \mathbb{R}^2$ *is said to be* quasiconformal *if each component of* f *belongs to* $W^{1,2}_{\text{loc}}(\mathbb{R}^2)$ *and if* f *satisfies the inequality*

$$|Df(x)|^2 \leq KJ_f(x) \text{ locally in } L^1(\mathbb{R}^2).$$

where $Df(x) = (\partial_i f_j)$ *and* $J_f = \det Df(x)$. **Note:** *The smallest such* K *is referred to as the* dilatation of f.

The locally in $L^1(\mathbb{R}^2)$ condition is deliberately used over an a.e. condition because, for instance, lines have measure zero and Cantor-Lebesgue type functions do not satisfy the integration by parts formula. It is not difficult to show that using definition (2) implies that any continuously differentiable quasiconformal map takes infinitesimal circles to infinitesimal ellipses of bounded eccentricity. Proving the converse is somewhat more involved.

Definition 3. *A Borel measure* μ *on* $\mathbb{R}^2$ *is a* doubling measure *if* μ *is non-trivial and there is a constant* C *(independent of* x *and* r*) so that* $\mu(B(x,r)) \leq C\mu(B(x,2r))$ *for any ball* $B(x,r)$ *centered at* x *of radius* r.

Any doubling measure μ may be associated with a quasimetric d_μ by the relation

$$d_\mu(x,y) = d_\mu(B_{xy}) = [\mu(B(x,|x-y|) \cup B(y,|x-y|))]^{1/2}.$$

We see that a typical d_μ only satisfies the following weak triangle inequality:

$$d_\mu(x,y) \leq K(d_\mu(x,z) + d_\mu(z,y)),$$

and so generally d_μ will not be a metric. This motivates the following definition:

Definition 4. *A* metric doubling measure μ *is a doubling measure for which the quasimetric* d_μ *defined above is comparable to a metric* δ_μ.

We may now observe that if $f : \mathbb{R}^2 \to \mathbb{R}^2$ is quasiconformal, then $\mu = J_f(x)dm_2(x)$ is a metric doubling measure, where m_2 is the standard Lebesgue measure on the plane.

Indeed, we may choose $\delta_\mu = |f(x) - f(y)|$. To see why this selection works, we'll need the following useful properties of quasiconformal maps (taken from [4]). Define $h(s) = \frac{(16(s+1/2))^K}{2}$. For any 3 points in the plane a, b, c, let T denote the triangle with those vertices and define $\text{skew}(a,b,c)$ to be the ratio of the longest side to the shortest side of T.

Theorem 5. *If $f : \mathbb{R}^2 \to \mathbb{R}^2$ is quasiconformal, then for any points x, y in the plane, there are neighborhoods D_x and D_y so that for any triples $a, b, c \subset D_x$ and $a', b', c' \subset D_y$, we have*

$$\begin{aligned} \operatorname{skew}(f(a), f(b), f(c)) &\leq h(\operatorname{skew}(a, b, c)) \\ \operatorname{skew}(f^{-1}(a'), f^{-1}(b'), f^{-1}(c')) &\leq h(\operatorname{skew}(a', b', c')). \end{aligned} \tag{1}$$

Conformal maps take infinitesimal equilateral triangles to equilateral triangles since those maps preserve angles. Inequality (1) says that quasiconformal maps can't do all that much worse. Moreover, inequality (1) allows us to prove that quasiconformal maps do not distort space too much.

First, we are able to prove that for all small equilateral triangles T, $\operatorname{diam}(f(T)) \lesssim m_2(f(T))^{1/2}$, where the constant depends only on how small we require the perimeter to be. The isoperimetric inequality gives the opposite inequality so that the two quantities are in fact comparable. A similar triangles argument can extend to arbitrary equilateral triangles T: $\operatorname{diam}(f(T)) \simeq m_2(f(T))^{1/2}$. It is not too hard to then extend the theorem to balls or other compact, convex sets. Second, one can show via inequality (1) that being K-quasiconformal is equivalent to the existence of a homeomorphism $\eta : [0, \infty) \to [0, \infty)$ with the property that

$$\left| \frac{f(x) - f(y)}{f(x) - f(z)} \right| \leq \eta \left(\left| \frac{x - y}{x - z} \right| \right).$$

Using this identity, it is not hard to show that if B_{xy} is defined as above,

$$|f(x) - f(y)| \simeq \operatorname{diam} f(B_{xy})).$$

Putting this discussion all together, we deduce that

$$\delta_\mu(x, y) \simeq \operatorname{diam} f(B_{xy})) \simeq m_2(f(B_{xy}))^{1/2} = \left(\int_{B_{xy}} J_f(x) dm_2(x) \right)^{1/2}.$$

Note that the right side is precisely $d_\mu(x, y)$, and so our choice of δ_μ works perfectly well.

David and Semmes proved in [2] that every metric doubling measure has a density in the so-called A_∞ class. At that time, it was already known that A_∞ included the quasiconformal Jacobians. David and Semmes called the class of densities of metric doubling measures strong-A_∞. It should come as no surprise that they asked whether the strong-A_∞ functions were all comparable to quasiconformal Jacobians. Unfortunately, Laasko showed the answer is negative [5].

It follows immediately from the properties used above, however, that if $f : \mathbb{R}^2 \to \mathbb{R}^2$ is quasiconformal, then for any metric δ comparable to d_μ we have $f : (\mathbb{R}^2, \delta) \to \mathbb{R}^2$ is a bi-Lipschitz equivalence. The converse to this question is affirmative. Specifically,

Theorem 6. *Suppose that μ is a metric doubling measure so that d_μ is comparable to a metric δ_μ. Any homeomorphism $f : (\mathbb{R}^2, \delta_\mu) \to \mathbb{R}^2$ which is bi-Lipschitz has the additional following properties:*

- *There is a non-negative function $w \in L^1_{\text{loc}}(\mathbb{R}^2)$ so that $d\mu = w dm_2$, where dm_2 is the standard Lebesgue measure on $\mathbb{R}^2$. In fact $w \in A_\infty$.*
- *$f : \mathbb{R}^2 \to \mathbb{R}^2$ is quasiconformal.*
- *$J_f \simeq w$ for a.e. $x \in \mathbb{R}^2$. Moreover, the comparability constant and the dilatation of f depend only on data associated with μ.*

The theorem follows quickly from results derived above.

7.2 Main Result

Theorem 6 creates a mechanism for producing quasiconformal Jacobians:

- Select a suitable (i.e. at least A_∞) function w.
- Show that $d\mu = w dm_2$ is a metric doubling measure with associated to some metric δ_μ.
- Produce a bi-Lipschitz equivalence $(\mathbb{R}^2, \delta_\mu) \to \mathbb{R}^2$.

We now state our main result.

Theorem 7. (Main Theorem) *Suppose that $u \in L^1_{\text{loc}}(\mathbb{R}^2)$ with distributional gradient $\nabla u \in L^2(\mathbb{R}^2)$. Then there is a quasiconformal map so that*

$$J_f(x) \simeq e^{2u(x)} \text{ for a.e. } x \in \mathbb{R}^2.$$

The comparability constant C and the dilatation of f depend only on $\|\nabla u\|_{L^2(\mathbb{R}^2)}$.

Proof of the Main Theorem. We will follow the usual proof strategy of first establishing the theorem is true for $u \in C_0^\infty(\mathbb{R}^2)$, then reducing the general case to this one by approximation. In the case of a smooth, compactly supported u, we have that $\exp(2u(x))$ is strictly positive and bounded above and below; hence showing that $d\mu = \exp(2u(x))dm_2$ is a doubling measure is easy. Likewise, showing that the quasimetric d_μ is comparable to $\exp(u(x))|dx|$ is trivial. The existence of a bi-Lipschitz equivalence is all that remains. This is of course easier said than done; however, our work is made substantially easier by the following theorem of Fu [3]:

Theorem 8. *Lex X be a complete Riemannian 2-manifold that is homeomorphic to $\mathbb{R}^2$. There are absolute constants $\epsilon_0 > 0$ and $L_0 > 0$ with the following property: if the integral curvature of X does not exceed ϵ_0 then X is bi-Lipschitz equivalent to $\mathbb{R}^2$ with bi-Lipschitz constant L_0.*

In this spirit, consider the Riemannian 2-manifold $X_u = (\mathbb{R}^2, e^{u(x)}|dx|)$. Because u is smooth and compactly supported, $\exp(u(x))$ is strictly positive, so the identity map is a homeomorphism. Observe that if $b \in C_0^\infty(\mathbb{R}^2)$, then $X_{u-b} = (\mathbb{R}^2, e^{u(x)-b(x)}|dx|)$ is A-bi-Lipschitz equivalent to $(\mathbb{R}^2, e^{u(x)}|dx|)$ by the identity map where $A = \exp(||b||_\infty)$.

Now, whenever we scale the Euclidean metric by a smooth, positive function h, we have the Liouville curvature equation:

$$\triangle(\log h) = -Kh^2$$

In our case, $h = e^u$; hence the Gaussian curvature of X_u is given by

$$K = -e^{-2s}\triangle u.$$

The total Gaussian curvature of X_u is then given by

$$\int_{X_s} |K|e^{2u}dm_2 = \int_{\mathbb{R}^2} |\triangle u| dm_2.$$

In order for Fu's theorem to be useful to us, we need to modify u to control the L^1-norm of $\triangle u$ without losing the bi-Lipschitz equivalence stated above. The following lemma – which follows entirely from classical theorems – gives us the power to choose b correctly so that $||u - b||_1$ is small.

Lemma 9. *Let $u : \mathbb{R}^n \to \mathbb{R}$ be a smooth function of compact support. Then for every $\epsilon > 0$ there exists a decomposition $u = s + b$ into two compactly supported smooth functions such that*

$$\|\triangle s\|_1 \leq \epsilon$$

and

$$\|b\|_\infty \leq \frac{13}{\epsilon}\|\nabla u\|_2^2.$$

Using lemma (9) pick a decomposition $u = s + b$ using $\epsilon = \epsilon_0/2$ where ϵ_0 is the absolute constant from Fu's theorem. The total Gaussian curvature of X_{u-b} is now smaller than $\epsilon_0/2$. By Fu's Theorem, X_{u-b} is bi-Lipschitz equivalent to $\mathbb{R}^2$. Thus X_u is bi-Lipschitz equivalent to the plane with coefficient

$$L_0 \exp(\frac{26}{\epsilon_0}\|\nabla u\|_2^2).$$

Hence for all compactly supported smooth functions, our theorem is true.

Now, suppose that u is locally integrable with $\nabla u \in L^2(\mathbb{R}^2)$. By [6] we may select a sequence $u_i \in C_0^\infty(\mathbb{R}^2)$ so that $u_i \to u$ in $L^2_{\text{loc}}(\mathbb{R}^2)$ and $\nabla u_i \to \nabla u$ in $L^2(\mathbb{R}^2)$. Then by the work above, we may find a sequence of K_i-quasiconformal maps $f_i : \mathbb{R}^2 \to \mathbb{R}^2$ so that

$$J_{f_i} \simeq \exp(2u_i). \tag{2}$$

Our choice of constant in (2) depends only on $e^{u_i(x)}dx$ and $\|\nabla u\|_2$. We shall show below that $\exp(2u_i)$ converges locally in $L^1(\mathbb{R}^2)$ to $\exp(2u)$ so that we may assume our constants of comparability are independent of i. For the same reason, each f_i will be K-quasiconformal for some sufficiently large (fixed) K.

Using Trudinger's inequality and Bunyakovsky's inequality we conclude that

$$\int_B e^{4v} dm_2 \leq ACm_2(B) \tag{3}$$

where B is an arbitrary ball in the plane. This in turn implies $\exp(2u_i) \to \exp(2u)$ in $L^1_{\text{loc}}(\mathbb{R}^2)$.

Now we proceed to show that some subsequence of f_i converges on all compact sets to a K-quasiconformal map f. Conjugation of f_i by a conformal mapping changes neither the Jacobian nor the dilatation. Thus we may assume without loss of generality that $f_i(0) = 0$ for all i. By [8], we conclude that (f_i) is a normal family and thus some subsequence converges locally uniformly to f. By [7], we deduce that f is K-quasiconformal, and, moreover, we have weak convergence of J_{f_i} to J_f. A calculation involving (2) and (3) allows us to deduce that J_f is comparable to $\exp(2u)$. The constant of comparability depends only on the constant of comparability from (2) and hence only on $\|\nabla u\|_2$. The same holds for the dilatation of f.

Thus our proof is complete. □

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8 Separated nets in Euclidean space and Jacobians of bi-Lipschitz maps

after C. McMullen [3] and D. Burago and B. Kleiner [1]
A summary written by John Maki

Abstract

Are all separated nets in $\mathbb{R}^n$ bi-Lipschitz equivalent to the integer lattice $\mathbb{Z}^n$? Also, is each non-negative $L^\infty(\mathbb{R}^n)$ function which is also bounded away from zero the Jacobian of a bi-Lipschitz mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$? We first show that the questions are equivalent, and then we show the common answer to both is no, demonstrated directly by a counterexample for the second question. Additionally, some related problems are considered.

8.1 Introduction

We begin with a definition.

Definition 1. *A subset X of a metric space Y is a separated net if there exist two constants a, $b > 0$ such that $d(x, x') > a$ for every pair x, $x' \in X$, and $d(y, X) < b$ for every $y \in Y$.*

We note two facts: Every metric space contains separated nets, and two spaces are quasi-isometric if and only if they contain bi-Lipschitz equivalent separated nets. If all separated nets in $\mathbb{R}^n$ are bi-Lipschitz equivalent to $\mathbb{Z}^n$, this simplifies the task of determining if two spaces are quasi-isometric; this clearly motivates the following question.

(Q1) Are all separated nets in $\mathbb{R}^n$ bi-Lipschitz equivalent to the integer lattice $\mathbb{Z}^n$?

This was posed in its present context by Gromov [2]. The answer to the corresponding question is yes if $\mathbb{R}^n$ is replaced by any non-amenable space (with slight conditions on local geometry) or by hyperbolic space $\mathbb{H}^n$ with $n \geq 2$. Also, it is easy to verify the answer is yes in $\mathbb{R}^1$.

A different question of some interest is that of attempting to prescribe the Jacobian of a homeomorphism. Specifically, for a given type of homeomorphism, can we identify a class of functions which coincides with the set of

Jacobians of those homeomorphisms? Recall Rademacher's theorem, which guarantees that a Lipschitz map is differentiable a.e. With this in mind, the particular version of this question which we consider is:

(Q2) Given a function $f \in L^\infty(\mathbb{R}^n)$ such that $\inf f > 0$, will f necessarily be the Jacobian of some bi-Lipschitz mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$?

Variations on this question include starting with Jacobians in L^∞ or Sobolev spaces [7], [8], or Hölder classes $C^{0,\alpha}$ [6].

8.2 Main results

The primary result of the two papers resolves (Q1) and (Q2). The first step of the proof shows the questions are equivalent, while the second answers the second question in the negative. Thus, we produce the desired theorems:

Theorem 2. *There exists a separated net in $\mathbb{R}^n$ which is not bi-Lipschitz equivalent to the integer lattice.*

Theorem 3. *Let $I := [0,1]$. Given $c > 0$, there is a continuous function $\rho : I^n \to [1, 1+c]$, such that there is no bi-Lipschitz map $f : I^n \to \mathbb{R}^n$ with*

$$\operatorname{Jac}(f) := \operatorname{Det}(Df) = \rho \quad a.e.$$

The answer to the latter theorem also resolves a similar question in quasiconformal mappings, namely that there is a continuous function which is not the Jacobian of a quasiconformal mapping in $\mathbb{R}^n$, since any quasiconformal mapping with Jacobian bounded away from zero and infinity must necessarily be bi-Lipschitz. In general, the effort to characterize the functions which correspond to Jacobians of such maps is called the *quasiconformal Jacobian problem* (see [4]; also, the subject of lectures by Peter Luthy). Notably, not equality but rather equivalence between the function and Jacobian is pursued in that program.

Similar in spirit is the work of Bonk and Kleiner in their description of properties which identify when a metric space will be quasisymmetrically equivalent to a 2-sphere (see [5]). Theorem 2 shows that a natural conjecture regarding the classification of subsets of $\mathbb{R}^n$ which are bi-Lipschitz equivalent to $\mathbb{Z}^n$ is in fact false.

8.2.1 The equivalence

In this section, we prove the equivalence of questions (Q1) and (Q2).

Theorem 4. *The following statements are equivalent:*

A. Every function $\rho \in L^\infty(\mathbb{R}^n)$ *such that* $\inf \rho > 0$ *is the Jacobian of some bi-Lipschitz mapping from* $\mathbb{R}^n$ *to* $\mathbb{R}^n$*.*

B. Every separated net $Y \subset \mathbb{R}^n$ *is bi-Lipschitz to* $\mathbb{Z}^n$*.*

Proof. (A) $\implies$ (B). Let $X \subset \mathbb{R}^n$ be a separated net. The task here is to match members of X to members of $\mathbb{Z}^n$, and the matching is ultimately guaranteed by set-theoretic machinery. To prepare for that endgame, we need to relate members of X with members of $\mathbb{Z}^n$ such that any subset of X relates to a subset of $\mathbb{Z}^n$ which is just as large (or larger), and vice versa. The bi-Lipschitz property will derive from choosing nearby members for the relations.

For each $y \in Y$, let C_y denote the Voronoi cell

$$C_y = \{x : |x - y| < |x - y'| \text{ for all } y' \text{ in } Y, y' \neq y\}.$$

Define

$$\rho(x) = \sum_{y\,:\,x \in \overline{C_y}} \frac{1}{\operatorname{vol} C_y}.$$

Since Y is separated, then $\inf \operatorname{vol} C_y > 0$, which means $\rho \in L^\infty(\mathbb{R}^n)$. Also, since Y is a net, then $\sup \operatorname{diam} C_y < \infty$, and we find $\inf \rho > 0$. Applying (A), we get a bi-Lipschitz homeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ with $\operatorname{Jac}(f) = \rho$. Denote $f(C_y)$ by D_y; note that $\operatorname{vol} D_y = 1$.

For $z \in \mathbb{Z}^n$, let E_z denote the unit cube centered at z. Define the relation $R \subset Y \times \mathbb{Z}^n$ as the set $R = \{(y, z) : y \in Y,\ z \in \mathbb{Z}^n, \text{ and } D_y \cap E_z \neq \emptyset\}$.

Let $A \subset Y$ be a finite subset and let $R(A) = \{z \in \mathbb{Z}^n : (y, z) \in R \text{ for some } y \in A\}$. Since the cubes labeled by $R(A)$ cover the cells D_y labeled by A and since $\operatorname{vol}(D_y) = \mathrm{E_z} = 1$, we get $|R(A)| \geq |A|$. Similarly, $|R^{-1}(B)| \geq |B|$ for any finite set $B \subset \mathbb{Z}^n$.

By the transfinite form of Hall's marriage theorem, R contains the graph of an injective map $\phi_1 : Y \to \mathbb{Z}^n$. Similarly, R^{-1} contains the graph of an injective map $\phi_2 : \mathbb{Z}^n \to Y$. The Schröder-Bernstein theorem guarantees the existence of a bijection in the presence of two such injective maps; let $\phi : Y \to \mathbb{Z}^n$ be this bijection.

Since $\operatorname{diam} D_y$ and $\operatorname{diam} E_z$ are bounded, then the distance $|f(y) - z|$ is also bounded for all $(y, z) \in R$. As $\phi(y)$ is chosen among the $z \in \mathbb{Z}^n$ for which $(y, z) \in R$, we get $\sup |\phi(y) - f(y)| < \infty$, and so the map $\phi : Y \to \mathbb{Z}^n$ is bi-Lipschitz, too.

(B) $\Longrightarrow$ (A). Let $\rho \in L^\infty(\mathbb{R}^2)$ with $\inf \rho > 0$. Let $I := [0, 1]$. Without loss of generality, we may assume $\rho : I^2 \to [1, 1 + c]$.

We begin with the set-up, which produces a separated net that allows us to approximate ρ in a particular way. Let $\{S_k\}_{k=1}^\infty$ be a disjoint collection of squares in $\mathbb{R}^2$ which have vertices with integer coordinates and sides which are parallel to coordinate axes. Further, we require that the side length l_k of S_k tends to infinity as $k \to \infty$. Let $\phi_k : I^2 \to S_k$ be the unique affine homeomorphism with scalar linear part, and define $\rho_k : S_k \to [1, 1 + c]$ by translating the values of ρ to S_k: $\rho_k := \rho \circ \phi_k^{-1}$.

Choose a sequence $m_k \in \mathbb{Z}^+$ with $\lim_{k\to\infty} m_k = \infty$ and $\lim_{k\to\infty} m_k/l_k = 0$. Subdivide S_k into m_k^2 equal subsquares of side length l_k/m_k. Let this collection be $\mathcal{T}_k = \{T_{ki}\}_{i=1}^{m_k^2}$. Finally, we subdivide each T_{ki} into n_{ki}^2 equal subsquares U_{kij} where n_{ki} is the integer part of $\sqrt{\int_{T_{ki}} \rho_k d\mathcal{L}}$. Note that the integral $\int_{U_{kij}} \rho d\mathcal{L}$ is approximately 1.

We now choose a separated net $X \subset \mathbb{R}^2$ by placing one point at the center of each square U_{kij} and one point at the center of each integer square not contained in $\cup S_k$. The points which are not in $\cup S_k$ are added to satisfy the definition of net; the "separated" requirement is naturally satisfied since we are choosing centers of integer squares throughout. The real business, however, will use the points of X inside of the S_k's.

By hypothesis, there is an L-bi-Lipschitz homeomorphism $g : X \to \mathbb{Z}^2$. Let $X_k = \phi_k^{-1}(X) \subset I^2$ and define $f_k : X_k \to \mathbb{R}^2$ by

$$f_k(x) = \frac{1}{l_k} \left(g \circ \phi_k(x) - g \circ \phi_k(x_k) \right)$$

where x_k is some basepoint chosen in X_k. The image of f_k lies in an $L \times L$ square which contains the origin, hence the f_k's are a uniformly bounded collection of L-bi-Lipschitz maps. From the proof of the Arzela-Ascoli theorem, there exists a subsequence of $\{f_k\}$ which converges uniformly to a bi-Lipschitz map $f : I^2 \to \mathbb{R}^2$.

The counting measure on X_k, normalized by $1/l_k^2$, converges weakly to ρ times the Lebesgue measure, since the subsquares of S_k were chosen to give X a "local average density" of approximately ρ_k^{-1} in S_k. The normalized counting measure on $f_k(X_k)$ converges weakly to Lebesgue measure, from which we can conclude $f_*(\rho\mathcal{L}) = \mathcal{L}|_{f(I^2)}$. Recalling that the area of the image of a set is equal to the integral of the Jacobian over this set, we get that $\rho = \mathrm{Jac}(f)$.

□

8.2.2 The answer is no

Proof of Theorem 3. For convenience, we will prove the theorem in the case of $n = 2$; the proof for higher values of n follows with only minor modifications.

Fix two constants L and $c > 1$, and let $I := [0, 1]$. We will construct a continuous function $\rho : I^2 \to [1, 1 + c]$ which is not the Jacobian of a bi-Lipschitz homeomorphism. Ultimately, we succeed in driving a wedge between ρ and such Jacobians using the following definition.

Definition 5. *We say that two points x, $y \in I^2$ are A-stretched under a map $f : I^2 \to \mathbb{R}^2$ if $d(f(x), f(y)) \geq Ad(x, y)$.*

We define a sequence of functions $\{\rho_N\}$ which force a self-improving stretching property. For $N \in \mathbb{N}$, let R_N be the rectangle $[0, 1] \times [0, 1/N]$ and define a "checkerboard" function $\rho_N : R_N \to [1, 1+c]$ by $\rho_N(x, y) = 1$ if $\lfloor Nx \rfloor$ is even and $1 + c$ otherwise.

Lemma 6. *There are $k > 0$, M, μ, and N_0 such that if $N \geq N_0$, $\epsilon \leq \mu/N^2$, then the following holds: if the pair of points $(0, 0)$ and $(1, 0)$ is A-stretched under an L-bi-Lipschitz map $f : R_N \to \mathbb{R}^2$ whose Jacobian differs from ρ_N on a set of area less than ϵ, then at least one pair of points of the form $((\frac{p}{NM}, \frac{s}{NM}), (\frac{q}{NM}, \frac{s}{NM}))$ is $(1 + k)A$-stretched, where p and q are integers between 0 and NM and s is an integer between 0 and M.*

Sketch of proof of Lemma 6. The proof is by contradiction: Assume that all such pairs of points are less than $(1+k)A$-stretched. Since the pair of points $(0, 0)$ and $(1, 0)$ are A-stretched, it is reasonable by the triangle inequality that most of the targeted pairs are approximately A-stretched.

Stating that more precisely, define subsquares $S_i := [\frac{i-1}{N}, \frac{i}{N}] \times [0, \frac{1}{N}]$ and points $x^i_{pq} := (\frac{p+M(i-1)}{NM}, \frac{q}{NM})$, where $i = 1, \ldots, N$, and p and q are integers between 0 and M. It can be shown that there is at least one square S_i which has the property that all of its points of the form x^i_{pq} are at least $(1-l)A$-stretched when paired with their corresponding neighbor point x^{i+1}_{pq}, for any $l \in (0,1)$.

Given the density of the points x^i_{pq} in S_i, the image of S_i under f is approximately a translate of the image of S_{i+1} under f, since we have bounded the relative stretch of these adjacent squares above and below by values close to A. However, ρ is very close to the Jacobian of f, which means that the areas of the images of S_i and S_{i+1} will be expanded by factors very close to 1 and $1+c$. This is in competition with the way in which the images of the sets are approximate translates of each other, and ultimately leads to a contradiction. □

We may use a smooth change of coordinates to derive the following lemma from Lemma 6.

Lemma 7. *There exists a constant $k > 0$ such that, given any segment $\overline{xy} \subset I^2$ and any neighborhood $\overline{xy} \subset U \subset I^2$, there is a measurable function $\rho : U \to [1, 1+c]$, $\epsilon > 0$ and a finite collection of non-intersecting segments $\overline{l_j r_j} \subset U$ with the following property: if the pair x, y is A-stretched by an L-bi-Lipschitz map $f : U \to \mathbb{R}^2$ whose Jacobian differs from ρ on a set of area $< \epsilon$, then for some j the pair l_j, r_j is $(1+k)A$-stretched by f. The function ρ may be chosen to have finite image.*

Finally, we see how this self-improving stretch can lead to a crisis for an L-bi-Lipschitz map, as we repeat the construction on smaller scales.

Lemma 8. *For each integer i there is a measurable function $\rho_i : I^2 \to [1, 1+c]$, a finite collection $\mathcal{S}_i$ of non-intersecting segments $\overline{l_j r_j} \subset I^2$, and $\epsilon_i > 0$ with the following property: For every L-bi-Lipschitz map $f : I^2 \to \mathbb{R}^2$ whose Jacobian differs from ρ_i on a set of area $< \epsilon_i$, at least one segment from $\mathcal{S}_i$ will have its endpoints $\frac{(1+k)^i}{L}$-stretched by f.*

Proof of Lemma 8. We proceed by induction. The case $i = 0$ is trivial. Assume that the lemma is true for $i-1$, where $\mathcal{S}_{i-1} = \{\overline{l_j r_j}\}$. Let $\{U_j\}$ be a disjoint collection of open sets with $\overline{l_j r_j} \subset U_j$ and with total area $< \epsilon_{i-1}/2$. For each j apply the previous lemma to U_j to get a function $\hat{\rho}_j : U_j \to [1, 1+c]$, $\hat{\epsilon}_j > 0$, and a disjoint collection $\hat{\mathcal{S}}_{i-1,j}$ of segments.

Now define $\rho_i : I^2 \to [1, 1+c]$ by $\rho_i(x) = \hat{\rho}_j(x)$ for $x \in U_j$ and $\rho_i(x) = \rho_{i-1}(x)$ otherwise. Let $\mathcal{S}_i = \cup \hat{\mathcal{S}}_{i-1,j}$ and $\epsilon_i = \min \hat{\epsilon}_j$. These will now exhibit the desired property. □

This lemma shows that certain measurable functions are not possible as the Jacobians of L-bi-Lipschitz maps, for a given fixed value of L, but it further shows that we may iterate at smaller scales to increase the range of L-values which will not work. As we desire $\rho = \mathrm{Jac}(f)$ a.e., the lemma will apply for all ϵ_i. Finally, we approximate the resulting sequence of measurable functions by a corresponding sequence of continuous functions, differing on a set of small measure tending to zero. The desired continuous function ρ will be the limit of this sequence of continuous functions. □

8.3 Related results

8.3.1 The answer for Hölder is yes

If the bi-Lipschitz requirement is relaxed to a homogeneous Hölder condition, we find the modified (Q1) and (Q2) have positive answers (see [3]).

Definition 9. *We say $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is a homogeneous Hölder map if there are constants $K \geq 0$ and $0 < \alpha \leq 1$ such that for $|x|, |y| \leq R$ we have*

$$|\phi(x) - \phi(y)| \leq KR^{1-\alpha}|x - y|^{\alpha}.$$

The purpose of this definition for Hölder is to make it a scale-invariant property: if $\phi(x)$ is homogeneous Hölder, then so is $c\phi(x/c)$ for any $c > 0$.

Theorem 10. *Fix $n \geq 1$. Then:*

1. *For any $f \in L^\infty(\mathbb{R}^n)$ with $\inf f > 0$, there is a homogeneous bi-Hölder homeomorphism $\phi : \mathbb{R}^n \to \mathbb{R}^n$ (i.e. ϕ^{-1} is also homogeneous Hölder) such that*

$$\mathrm{vol}(\phi(E)) = \int_E f(x)dx$$

   for all bounded open sets $E \subset \mathbb{R}^n$.

2. *For any separated net $Y \subset \mathbb{R}^n$, there is a homogeneous bi-Hölder bijection $\psi : Y \to \mathbb{Z}^n$.*

8.3.2 Prescribing the divergence of a vector field

An infinitesimal form of the problem of constructing a map with prescribed volume distortion is to consider whether it is possible to prescribe the divergence of a Lipschitz or quasiconformal vector field v using a function $f \in L^\infty(\mathbb{R}^n)$. We have the following result from [3]:

Theorem 11. *For any $n > 1$ there is an $f \in L^\infty(\mathbb{R}^n)$ which is not the divergence of any Lipschitz, or even quasiconformal, vector field.*

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9 Newtonian spaces: An extension of Sobolev spaces to metric measure spaces

after N. Shanmugalingam [3]
A summary written by Kenneth Maples

Abstract

We extend the definition of Sobolev spaces to metric measure spaces and prove corresponding results to the classical embedding theorems.

9.1 Introduction

We would like to extend the definition of Sobolev spaces ($W^{1,p}$) to more general spaces than $\mathbb{R}^n$ such as Riemannian manifolds. This is so that we can define PDE in greater generality. However, the definition of Sobolev space depends on the distributional derivatives of the function; this concept may not be defined on a general metric measure space. Tthe definition of Sobolev space must be replaced by an equivalent definition that only depends on the metric and measure of the underlying space.

A previous attempt by Hajłasz noted that $u \in W^{1,p}$ if and only if there was a p-integrable $g \geq 0$ such that

$$|u(x) - u(y)| \leq |x - y|(g(x) + g(y))$$

for almost all x, $y \in \mathbb{R}^n$. In [3] Shanmugalingam replaces this inequality with one based on "upper gradients" of the function, which generalizes the idea of a primitive of a vector field. Instead, we let $u \in \widetilde{N}^{1,p}$ if there is a p-integrable $\rho \geq 0$ such that

$$|u(x) - u(y)| \leq \int_\gamma \rho$$

for all rectifiable compact paths γ except for a set of paths of "p-modulus zero."

This paper defines the "Newtonian" (also known as "Newton-Sobolev") spaces $N^{1,p}$ and proves several fundamental theorems about them. Notably, $N^{1,p}$ is shown to be a complete subspace of L^p, and several analogues of classical Sobolev embedding theorems are proven. For a motivation of the techniques used see [2].

9.2 Preliminaries

Let (X, d, μ) denote a space X with metric d and measure μ. Let $\mathcal{H}_1$ denote Hausdorff one-dimensional measure. A *path* is a continuous map $\gamma : I \to X$ where I is some interval in $\mathbb{R}$. The image is denoted by $|\gamma|$ and the length is denoted by $l(\gamma)$. We have the following standard notations for path families:

$\Gamma_{\mathbf{rect}}$ Non-constant rectifiable paths with compact image (equivalently I is compact).

Γ_A Paths in Γ_{rect} that intersect the set A.

Γ_A^+ Paths in Γ_{rect} that intersect the set A on a set of *positive* one-dimensional Hausdorff measure; i.e. $\mathcal{H}_1(|\gamma| \cap A) > 0$.

We will only consider subfamilies of Γ_{rect}; we will therefore assume that paths are parameterized by arc-length.

We define the p-modulus of a family of paths Γ as follows:

$$\text{Mod}_p\Gamma = \inf_{\rho \wedge \Gamma} \|\rho\|_{L^p}^p$$

We write $\rho \wedge \Gamma$ if ρ is "admissible" for the family Γ; this means that $\rho \geq 0$ is Borel-measurable and $\int_\gamma \rho \, dx \geq 1$ for all $\gamma \in \Gamma$.

There are several general properties of p-modulus that we will omit here; a general (and highly readable) treatment is in [1].

If Q is a statement about paths, then we say that Q holds p-a.e. if the family of paths where Q is *false* has zero p-modulus.

A function u is called ACC_p if it is absolutely continuous on p-a.e. path. In other words, $u \circ \gamma$ is absolutely continuous on $[0, l(\gamma)]$ for p-a.e. $\gamma \in \Gamma_{\text{rect}}$. Likewise, if X is a domain in $\mathbb{R}^n$, we say that a function u is ACL if it is absolutely continuous on every line perpendicular to the axes; in other words, if $u \circ \gamma$ is absolutely continuous on $[0, l(\gamma)]$ for $\mathcal{H}_{n-1}$-a.e. line γ parallel to the coordinate axes. Note that an ACL function has directional derivatives almost everywhere. We call an ACL function ACL_p if the directional derivatives are in L^p.

Given $u : X \to \mathbb{R}$, we call a Borel-measurable $\rho \geq 0$ an *upper gradient* of u if for all $\gamma \in \Gamma_{\text{rect}}$,

$$|u(x) - u(y)| \leq \int_\gamma \rho \, ds$$

where x and y are the endpoints of γ. We will also write $\rho \gg u$ (or $u \ll \rho$) for this. If the inequality only holds on p-a.e. path, then ρ is called a *p-weak upper gradient* of u and we will write $\rho \gg_p u$ (or $u \ll_p \rho$).

We say that X supports a $(1,p)$-*Poincaré inequality* if there exists a uniform constant $C > 0$ such that for all open balls B in X and pairs of functions (u, g) with $u \ll g$ and $u \in L^1$,

$$\frac{1}{\mu(B)} \int_B |u - u_B| \leq C \operatorname{diam}(B) \left(\frac{1}{\mu(B)} \int_B g^p \right)^{1/p}.$$

If X supports a $(1,p)$-Poincaré inequality, then it supports a $(1,q)$-Poincaré inequality for all $q > p$.

9.3 $N^{1,p}$ and friends

Let Ω be a domain in $\mathbb{R}^n$. For $1 \leq p < \infty$ we can define the Sobolev space $W^{1,p}$ as the space of functions on Ω with finite $W^{1,p}$ norm, where

$$\|u\|_{W^{1,p}} = \|u\|_{L^p} + \sum_{k=1}^{n} \|\partial_i u\|_{L^p}$$

Here ∂_i is the distributional derivative in the ith coordinate.

The goal of this paper is to extend the definition of Sobolev spaces to more general spaces than $\mathbb{R}^n$. One earlier attempt noted that $u \in W^{1,p}$ if and only if there exists some $g \geq 0$, $g \in L^p$ such that for a.e. x, $y \in \mathbb{R}^n$,

$$|u(x) - u(y)| \leq |x - y|(g(x) + g(y))$$

We will call such functions g a *Hajłasz gradient* of u. We define $M^{1,p}(X)$ to be functions $u \in L^p$ along with a $g \geq 0$, $g \in L^p$ such that for μ-a.e. $x, y \in X$,

$$|u(x) - u(y)| \leq d(x,y)(g(x) + g(y))$$

This space carries the norm:

$$\|u\|_{M^{1,p}} = \|u\|_{L^p} + \inf_g \|g\|_{L^p}$$

where the infimum is taken over all Hajłasz gradients of u. $M^{1,p}(X)$ is Banach with this norm.

The space introduced in this paper exploits the concept of upper gradients, defined in the first section. Let $\widetilde{N}^{1,p}(X)$ be the space of all $u \in L^p(X, \mathbb{R})$ such that $u \ll_p g$ for some $g \geq 0$, $g \in L^p$. $\widetilde{N}^{1,p}$ is a vector space; it is also a lattice. We can define a seminorm on $\widetilde{N}^{1,p}$ in the following way:

$$\|u\|_{\widetilde{N}^{1,p}} = \|u\|_{L^p} + \inf_{u \ll_p g} \|g\|_{L^p}$$

This seminorm is *not* positive-definite. However, the relation $u \sim v$ if $\|u - v\|_{\widetilde{N}^{1,p}} = 0$ is an equivalence relation and partitions $\widetilde{N}^{1,p}$ into equivalence classes. We will denote the space of these equivalence classes by $N^{1,p}(X)$ with the norm induced by the seminorm above.

Another extension of Sobolev spaces based on upper gradients was developed by Cheeger. We define $H_{1,p}(X)$ to be the subspace of L^p such that

$$|f|_{1,p} = \|f\|_{L^p} + \inf_{\{f_i\}} \liminf_{i \to \infty} \|g_i\|_{L^p}$$

is finite. The limit infimum is taken over all upper gradients g_i of the functions (f_i) where $(f_i) \to f$ in L^p.

9.4 $N^{1,p}$ is Banach

To show that $N^{1,p}$ is complete, we must introduce the capacity of a set. We can define the p-capacity of a *set* $E \subseteq X$ as:

$$\mathrm{Cap}_p E = \inf_{u \wedge E} \|u\|^p_{N^{1,p}}$$

where $u \wedge E$ means that u is "admissible" for the set E; i.e. $u \in N^{1,p}$ such that $u \geq 1$ on E. Other definitions for the capacity of a set are possible, but the differences appear to be immaterial.

The following lemma is key to the argument:

Lemma 1. *If $F \subseteq X$ with $Cap_p F = 0$, then $Mod_p \Gamma_F = 0$.*

Theorem 2. *$N^{1,p}$ is Banach.*

Proof. We follow the same procedure as for showing that L^p is complete. We choose a Cauchy sequence (u_i) in $N^{1,p}(X)$ and, after noting that it suffices to examine subsequences, select a subsequence with rapid convergence. We can force upper gradients of the difference of consecutive terms (i.e. $g_{k,k+1} \gg (u_k - u_{k+1})$) to have similarly shrinking L^p norm. We then construct

$$E_k = \{x \in X : |u_k(x) - u_{k+1}(x)| \geq 2^{-k}\}.$$

As (E_k) has capacity rapidly converging to 0, we also construct $F = \limsup E_k$. On $X \setminus F$, (u_k) converges pointwise to a function u. By the previous lemma, because $\mathrm{Cap}_p F = 0$, $\mathrm{Mod}_p \Gamma_F = 0$. Hence on every $\Gamma_{\mathrm{rect}} \setminus \Gamma_F$ we can sum up the difference upper gradients $g_{k,k+1}$ to form an upper gradient of $u - u_k$ for each k. This gives an upper gradient for u and similarly shows that $(u_k) \to u$ in $N^{1,p}$. □

9.5 Similarities between $N^{1,p}$ and $M^{1,p}$

First we verify that $N^{1,p}(X)$ is a genuine extension to $W^{1,p}$.

Theorem 3. *If $X = \Omega$ is a domain in $\mathbb{R}^n$, $d(x,y) = |x - y|$, and μ is Lebesgue n-measure, then $N^{1,p}(X) = W^{1,p}(\Omega)$ as Banach spaces.*

Proof. This theorem follows from considering the absolute continuity properties of $u \in N^{1,p}$. □

Lemma 4. *The set of equivalence classes of continuous functions $u \in M^{1,p}$ embeds into $N^{1,p}$ with*

$$\|u\|_{N^{1,p}} \leq 4\|u\|_{M^{1,p}}$$

Proof. We can choose u to have a Hajłasz gradient g such that the Hajłasz inequality holds everywhere. If γ connects $x, y \in X$, consider $\int_\gamma g$. If the integral is ∞ then the upper gradient inequality holds. If it is finite, we can partition γ into n pieces γ_i of equal length. Choose $x_i \in |\gamma_i|$ such that $g(x_i) \leq \mathcal{H}_1(|\gamma_i|)^{-1} \int_{\gamma_i} g$; then we can convert the Hajłasz inequality into the upper gradient inequality:

$$\begin{aligned} |u(x_0) - u(x_n)| &\leq \sum d(x_i, x_{i+1})(g(x_i) + g(x_{i+1})) \\ &\leq 4 \int_\gamma g \end{aligned}$$

Letting $n \to \infty$ we have the upper gradient inequality for the endpoints of γ. The embedding is well-defined. □

From this lemma,

Theorem 5. *The Hajłasz space $M^{1,p}(X)$ continuously embeds into the $N^{1,p}(X)$.*

Theorem 6. *If X is a doubling space and X supports a $(1,q)$-Poincaré inequality for some $q \in (1,p)$, then $N^{1,p}(X) = M^{1,p}(X)$ isomorphically as Banach spaces.*

Finally, the two definitions of Sobolev-type spaces based on upper gradients are nearly the same, as the following theorem shows.

Theorem 7. *$H_{1,p}(X)$ is isometrically equivalent to $N^{1,p}(X)$ when $p > 1$.*

9.6 Analogues of Sobolev space results

Theorem 8. *If X is a doubling space that supports a $(1,p)$-Poincaré inequality, then Lipschitz functions are dense in $N^{1,p}(X)$.*

Recall the following classical embeddings for $W^{1,p}(\mathbb{R}^n)$:

$$W^{1,p}(X) \hookrightarrow L^{np/(n-p)}, \qquad \text{if } p < n,$$

$$W^{1,p}(X) \hookrightarrow C^{0,1-n/p}, \qquad \text{if } p > n,$$

Theorem 9. *Let $Q > 0$. If X is a doubling space satisfying*

$$\mu(B(x,r)) \geq Cr^Q$$

with C uniform over x, $0 < r < 2 \operatorname{diam} X$, and if X supports a $(1,p)$-Poincaré inequality for some $p > Q$, then $N^{1,p}(X)$ continuously embeds into the space $C^{0,1-Q/p}$.

Equivalently, every equivalence class in $N^{1,p}$ has a Hölder continuous representative with exponent $1 - Q/p$ with bounded Hölder norm.

Proof. We define two sequences of balls that converge to x and y, respectively, with initial radii $d(x,y)$ such that the radii shrink in half at each step. If x and y are Lebesgue points of u, then the average values of u on the balls converges to the function values; after some calculation, we derive

$$|u(x) - u(y)| \leq C(Q,p)\|\rho\|_{L^p} d(x,y)^{1-Q/p}.$$

Because μ is doubling, the non-Lebesgue points form a set of measure zero. Therefore, u restricted to the Lebesgue points is Hölder continuous and can be extended to a Hölder continuous function $\widetilde{u}$ on all of X. By excluding the zero p-modulus families Γ_L^+ and $\Gamma_0 = \{\gamma : u \circ \gamma \text{ is } \textit{not} \text{ absolutely continuous}\}$, we have $u = \widetilde{u}$ on p-a.e. path, hence $\widetilde{u} \in N^{1,p}$ of the same equivalence class as u. □

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10 Two counterexamples in the plane

after C. Bishop [1] and T. Laakso [2]
A summary written by William Meyerson

Abstract

We equip the standard Euclidean plane with two pathological weights: an A_1 weight which is not comparable to a quasiconformal Jacobian and a strong A_∞ weight whose induced metric is not biLipschitz equivalent to Euclidean distance.

10.1 Introduction

Definition 1. *An A_1 weight on some Euclidean space E is a locally integrable $w : E \to [0, \infty]$ such that whenever B is a ball, the average value of w on B is bounded (up to a multiplicative constant C independent of B) by its essential infimum on B.*

Definition 2. *A strong A_∞ weight on E is a locally integrable $w : E \to [0, \infty]$ such that the measure induced by $\int w dx$ (where dx is Lebesgue measure) is doubling and the distance function sending a pair (x, y) to the integral of w over the ball centered at $(x + y)/2$ with radius $(x - y)/2$ is biLipschitz equivalent to some metric.*

If w is A_1 then clearly $w \preceq Mw$ almost everywhere (where M refers to the Hardy-Littlewood maximal function) by Lebesgue differentiation; however, because we need only consider those balls with rational centers and radii when looking at the maximal function and for such a ball,the average value of w on B is bounded above by Cw almost everywhere, we therefore have that $Mw \preceq w$ almost everywhere as well.

Further, A_1 weights are well-known by a straightforward geometric argument to be strong A_∞ weights; by a 1993 theorem of Semmes [3], E (equipped with the distance function induced by some A_1 weight) is biLipschitz equivalent to a subset of some larger Euclidean space.

However, as will be shown below by a counterexample due to Laakso [2] in the plane, this does not hold for strong A_∞ weights.

Further, we shall also introduce a reasonable conditions which strong A_∞ maps (and even A_1 maps) do not necessarily satisfy, even in $\mathbb{R}^2$. To do this, we need more definitions:

Definition 3. *A continuous function $f : \mathbb{R}^2 \to \mathbb{R}^2$ is said to be quasiconformal if there exists some constant C such that for each $x \in \mathbb{R}^2$, the limit superior of $max_{|z-x|=r,|y-x|=r} \frac{|f(z)-f(x)|}{|f(y)-f(x)|}$ as r goes to zero is bounded above by C.*

Remark 4. *Note that this is a generaliztion of conformal maps in BBR^2 (for these maps, $C = 1$ and the theory is well-known as conformal maps of the plane can be viewed as complex-valued functions; in this setting they are either analyticor conjugate-analytic).*

Definition 5. *We say that a strong A_∞ w is comparable to a quasiconformal Jacobian if there exists some quasiconformal f from $\mathbb{R}^2$ to $\mathbb{R}^2$ such that $Jf \sim w$ a.e. (where Jf is the Jacobian of f).*

With all the machinery set up, we can finally state the following two results:

Theorem 6. *(Laakso, [2]) There exist a strong A_∞ weight w on $\mathbb{R}^2$ which is not comparable to a quasiconformal Jacobian.*

Theorem 7. *(Bishop, [1]) There exist an A_1 weight w on $\mathbb{R}^2$ whose induced metric is not biLipschitz embeddable in any Euclidean space and therefore w is not comparable to a quasiconformal Jacobian.*

10.2 Proof of our first theorem

We shall proceed (following [1]) via Laakso's construction of a strong A_∞ weight in $\mathbb{R}^2$ such that when $\mathbb{R}^2$ is given the induced distance metric, it is not bi-Lipschitz embeddable to any Euclidean space. In doing so, we shall assume the following 1996 theorem of Semmes [4]:

Theorem 8. *Suppose that (X, d) is a metric space and $0 < t < 1$ such that the snowflake space (X, d^t) can be embedded biLipschitzly into some Euclidean space E. Then there exists a strong A_∞ weight w in E such that (X, d) can be embedded biLipschitzly into E equipped with the induced metric from w.*

Fixing $a \in (3/4, 1)$ and $k \in (0, \pi/6)$ there exists some $b \in (0, 1/4)$ such that $a + be^{ik} = 1 - be^{-ik}$; we define the five similarity mappings $S_1, \ldots, S_5$ on the Euclidean plane as follows: $S_1(z) = az$, $S_2(z) = a + be^{ik}z$, $S_3(z) = a - be^{ik}z$, $S_4(z) = 1 - be^{ik}z$, and $S_5(z) = 1 - be^{-ik}z$.

Our planar set Z shall be defined as the closure of the images of 0 under repeated iterations of the similarity mappings; as $0 = S_1(0)$ we have that Z is the union of its first order similarity parts $S_1Z, \ldots, S_5Z$ (here we say that an nth order similarity part is the image of Z under n mappings in $S_1, \ldots, S_5$. Clearly S_1Z intersects S_2Z and S_3Z at a, while S_4Z and S_5Z intersect at 1; further, S_2Z and S_4Z intersect at $a + be^{ik}$ while S_3Z and S_5Z intersect at $a - be^{ik}$.

To show that these are the only intersection and boundary points of first order pieces one notes (this requires a bit of a geometric argument, which we omit here) that there exists a minimal angle θ such that if $\{S_jZ, S_kZ\}$ is a pair of similarity parts whose intersection point z was listed above, and $x \neq z, y \neq z$ are in S_jZ, S_kZ respectively then the angle xzy has argument at least ρ.

We next choose some number $s > 1$ such that $a^s + 2b^s = 1$ (note here $1 = a + 2b\cos k < a + 2b$). Letting Z_n be the set of boundary points of nth order similarity parts (i.e. the elements achieved when applying n similarity transforms to either 0 or 1) we can define a metric d_n on G_n as follows:

$d_n(x, y)$ is the infimum of $|a_2 - a_1|^s + \cdots + |a_m - a_{m-1}|^s$ where the $\{a_i\}$ are a sequence of adjacent (i.e. any two consecutive a_i form the boundary points for some nth order similarity part) elements of Z_n such that $a_1 = x$ and $a_m = y$.

Clearly this is a metric and $Z_n \supset Z_k$ for each $n > k$. To define a metric on the union of the Z_n's (whose closure is Z itself) it therefore suffices to show that d_{n+1} is an extension of d_n for each n.

To see this we note that if $x = F(0)$ and $y = F(1)$ are the boundary points of an nth order similarity part FZ (corresponding to zero and one, respectively, under the appropriate composition F of similarities) then $d_{n+1}(x, y) = |x - y|^s$ (we can see this by noting that an optimal path goes from $F(0)$ to $F(a)$ to $F(a + be^{ik})$ to $F(1)$; considering Z to be a zeroth order similarity part this argument even works for 0 and 1).

Consequently, if x and y lie in Z_n then an optimal chain of adjacent boundary points in Z_{n+1} joining x and y can be made by taking the optimal chain of boundary points in Z_n from x to y and between each consecutive pair of points in this chain (which can be written as $F(0)$ and $F(1)$ for some nth order similarity) inserting the subsequence $\{F(a), F(a + be^{ik})\}$. This chain therefore witnesses that $d_{n+1}(x, y) = d_n(x, y)$ so we can indeed define d to be the union of the d_n on the union of the Z_n.

To extend d to a metric on all of Z it suffices to show that d is continuous with respect to Euclidean distance $|.|$; we shall do this by showing that d is biLipschitz equivalent to the quasinorm $|.|^s$.

To do this first note that if $x \notin S_1 Z$ then $d(x, 0)$ and $|x - 0|^s$ are both in $[a^s, 1]$ so d and the quasinorm are comparable for these points; by applying the inverse map of S_1 to an arbitrary nonzero $x \in Z$ we get the same comparability (i.e. $d(x, 0) \sim |x - 0|^s$). Similarly, if $y \notin S_4 Z \cup S_5 Z$ then $d(y, 1) \sim |y - 1|^s$; iterating inverse images of S_4 and S_5 gives the same result for arbitrary y different from 1.

If x is a boundary point of some similarity part FZ and y is an interior point of FZ then applying F^{-1} and the preceding paragraph yields $d(x, y) \sim |x - y|^s$ (as $F^{-1}(x)$ is zero or one).

Further, if x and y are in different similarity parts which do **not** share a boundary point, then as $d(x, y)$ and $|x - y|^s$ both lie in $((b/2)^s, 1]$, $d(x, y) \sim |x - y|^s$.

If x and y lie in adjacent similarity parts with common boundary point z then as $|x - y| \sim |x - z| + |z - y|$ (because of the lower bound on the angle xzy) so $|x - y|^s \sim |x - z|^s + |z - y|^s$ and $d(x, y) \sim d(x, z) + d(z, y)$, $d(x, y) \sim |x - y|^s$.

Finally, if x and y lie in the same similarity part, then by repeatedly applying inverse similarity transformations we can reduce to an earlier case so no matter what, $d(x, y) \sim |x - y|^s$ as desired; we can therefore suppose d is defined on all of Z.

To show that Z (equipped with the metric d) does **not** embed biLipschitzly into any Euclidean space, we shall use the following 'rounded ball' property of Euclidean spaces (which is an easy consequence of the parallelogram law for Hilbert spaces):

Proposition 9. *For each $\epsilon > 0$ there exists $\delta > 0$ such that if $x \neq y$ in some Euclidean space E then $B(x, |x - y|(1 + \delta)/2) \cap B(y, |x - y|(1 + \delta)/2)$ has diameter less than $\epsilon|x - y|$.*

Consequently we suppose for a contradiction that $f : (Z, d) \to E$ is biLipschitz (with coefficient L) where E is Euclidean. For any unequal pair of points $x, y \in Z$ we use $A_{x,y}$ to refer to $\frac{|f(x)-f(y)|}{d(x,y)}$; $\frac{1}{L} \leq A_{x,y} \leq L$. Letting $x = a + be^{ik}$ and $y = a + be^{-ik}$, we have that $d(a, x) = d(x, 1) = d(1, y) = d(y, a) = b^s$ while $d(a, 1) = d(x, y) = 2b^s$.

Therefore $|f(x) - f(y)| \geq L^{-2}|f(a) - f(1)|$ so by the rounded ball property, if $f(x)$ and $f(y)$ are both within r of $f(a)$ and $f(1)$ then there exists $\delta > 0$

such that $r > (1+\delta)|f(a) - f(1)|/2$. This implies that either $A_{1,x}$, $A_{1,y}$, $A_{a,x}$, or $A_{a,y}$ (all of which are boundary points of a similarity part) is at least $(1+\delta)A_{a,1}$.

We derive a contradiction as follows: because $A_{0,1} \geq \frac{1}{L}$ there exists $n \geq 0$ such that $A_{a^n,a^{n+1}} \geq \frac{1}{L}$ which implies that there exist some pair (s,t) of boundary points of a similarity part (one will be either a^n or a^{n+1}; the other will be either $a^n x$ or $a^n y$) with $A_{s,t} \geq \frac{(1+\delta)}{L}$. By repeating this argument for this similarity part (applying the appropriate tranformation) we get a pair (s',t') of boundary points of a similarity part contained in the similarity part bounded by s and t such that $A_{s',t'} \geq \frac{(1+\delta)^2}{L}$. Iterating this procedure recursively implies that the Lipschitz coefficient of f is at least $\frac{(1+\delta)^n}{L}$ for each n; letting n go to infinity contradicts that f is Lipschitz. By our theorem of Semmes there exists a strong A_∞ weight w on $\mathbf{R}^2$ such that (Z,d) is bi-Lipschitz equivalent to a subset of $\mathbf{R}^2$ with distance metric induced by w.

To prove our final result we now supose f is a quasiconformal (and therefore quasisymmetric) map from $\mathbb{R}^2$ to itself such that w is comparable to Jf; further, we use m to denote Lebesgue measure. If x and y are unequal points in $\mathbf{R}^2$ we use $B_{x,y}$ to denote the ball centered at $(x+y)/2$ with radius $|x-y|/2$; therefore quasisymmetry gives that $m(f(B_{x,y})) \sim |f(x) - f(y)|^2$. As f is absolutely continuous (and the standard change of variables rules apply) we note that letting w be the volume derivate of f (i.e. the limit of $\frac{m(f(B(x,r)))}{m(B(x,r))}$, which equals $|Jf|$ ae) yields that

$$\int \chi_{B_{x,y}}(z)w(z)dz = \int \chi_{B_{x,y}}(f^{-1}(a))w(f^{-1}(a))Jf^{-1}(a)da$$

$$= \int \chi_{f(B_{x,y})}(z)\frac{w(a)}{w(a)}da = m(f(B_{x,y}))$$

so $|f(x) - f(y)| \sim d(x,y)$ where d is the metric induced by w, producing a contradiction (as f is therefore biLipschitz) and proving the theorem as desired.

10.3 Proof of our second theorem

Following [1], we begin by constructing a descending sequence $E_0 \supset E_1 \supset \ldots E_n \ldots$ of subsets of the unit square $[0,1]$ x $[0,1]$ in the plane; their intersection E will be a set of measure zero on which our desired weight should be infinite.

To do this one needs to introduce the notion of (L, M, N)-pieces where L, M, N are positive integers such that L, N are odd and greater than or equal to three.

Definition 10. *In this setting an (L, M, N) piece of a square Q is the subset produced as follows: we first divide Q into L^2 equal subsquares (to be called the type 1 subsquares of Q) with center square Q_0; next we divide Q_0 into M^2 equal subsquares (to be called the type 2 subsquares of Q) and finally divide these type 2 subsquares into N^2 equal subsquares (to be called the type 3 subsquares); the (L, M, N) piece of Q is defined to be equal to Q with the (open) central type 3 subsquare of each type 1 subsquare removed.*

Supposing that $\{L_n\}$, $\{M_n\}$, and $\{N_n\}$ are sequences of this type, we construct a sequence $\{E_n\}$ of compact sets as follows (for notational purposes we let $t_n = (L_n M_n N_n)^{-1}$ and $s_n = t_1 * \cdots * t_n$):

We begin by letting E_0 be the unit square; for $n \geq 1$ we take each partition E_{n-1} into disjoint squares of size s_{n-1} (these are the size of the type 3 subsquares; we call them the $(n-1)$ generation squares) and replace each such square with the corresponding (L_n, M_n, N_n)-piece.

For the remainder of this talk we assume that

$$\Sigma N_n^{-2} = \infty, \Sigma N_n^{-3} < \infty, \Sigma M_n^{-2} < \infty$$

while the L_n are constant; for concreteness we can set $L_n = 7$, $M_n = n$, $N_n = 3 + 2\lfloor \frac{\sqrt{n}}{16} \rfloor$.

From these assumptions we know that area of E is zero; in fact, if S is an nth-generation square and $k > n$ then $|E_k \cap S| \preceq \frac{n}{k}|S|$.

Further, if Q is an nth generation square (of side length s_n) the second-outermost layer of type 1 subsquares are of size $s_n/7$. This layer forms a ring separating the central type 1 subsquare Q' of Q from the boundary of Q; also, any point on this ring is at least $s_n/7$ away from both Q' and δQ.

Definition 11. *In this setting, we call these squares the ring squares W_Q of Q.*

By the fact that quasiconformal maps on $\mathbf{R}^n$ are quasi-symmetric, we have that if f is K-quasiconformal and fixes 0 and ∞,

$$d(f(W), \delta f(Q) \cup f(Q')) \preceq_K diam(f(Q)).$$

Armed with these sets, we can now construct our weight w. We begin by setting F_n to be a s_n-neighborhood of E_n for each n; this serves to fill in the type 3 subsquares removed from E_{n-1} to get to E_n and adding (if necessary) a ring of size s_n around each $n-1$st generation square. Clearly this comes nowhere close to doubling the area of E_n; further, $d(F_n, F_{n-1}^c) \geq \frac{1}{2}s_{n-1}$ because the points in F_n are no more than s_n away from E_{n-1} in horizontal and vertical distance.

Letting $\{A_n\}$ be an increasing sequence of positive numbers such that $A_0 = 1$ and A_n approaches infinity with n we set $w = 1$ outside of F_1 and $w = A_n$ on $F_n \backslash F_{n+1}$; further, for $n \geq 1$ we can set $a_n = A_n - A_{n-1}$.

Making the additional assumption that there exists some $\lambda < \sqrt{2}$ so that $A_{2n} \leq \lambda A_n$ (which implies $A_n = o(\sqrt{n})$) and the bookkeeping assumption that $a_n \leq 1$ for each n we bound the average value of w on some nth generation square S as follows (this will imply local integrability) noting that this average is (up to a constant) bounded above by $n\Sigma_{k=n}^{\infty}\frac{a_k}{k}$: we have

$$n\Sigma_{k=n}^{N}\frac{a_k}{k} \leq O(A_n) + n\frac{A_N}{N}$$

by a computation which implies (letting N go to infinity) that the average value of w over a dyadic square S is bounded above (up to a multiplicative constant) by its essential infimum w on that square.

To show that w is A_1 we therefore need only consider a ball $B = B(x, r)$ of radius r; letting n be the largest positive integer with $s_n > 4r$ we let k be such that A_k is the infimum of w on B (we can do this because w is infinite only on a set of measure zero); if $k \leq n$ then as $4r \leq s_k$ we have that $2B$ is outside E_{k+1} (as points within $4r$ of E_{k+1} lie in F_{k+1}) and therefore B lies outside F_{k+2} (otherwise we could move by $s_{n+2} << r$ to get a point in E_{k+2} which lies in $2B$) so w is bounded above by $A_{k+2} \leq 3A_k$ on B.

However, if $k > n$ then as $r \geq \frac{1}{4}s_k$ we can cover B by disjoint $k+1$-generation squares (of size s_{k+1}) which are all contained in $2B$; therefore, the average of w over B is at most four times the largest average of w over one of these square; which is bounded above by (up to a multiplicative constant) $4A_{k+1} \leq 8A_k$ and therefore w is A_1.

Now, if x is not in E_n then x is as far from E_n as from E (because edges of nth generation squares in E_n are never removed in any later stage). Also, if x is in $E_n \backslash E$ then x was removed at the kth stage for some $k > n$; therefore, x is at most $s_k/2 \leq s_{n+1}/2$ from E_k (and thus is that distance away from E) so

$$\{x : s_{n+1} \leq d(x, E) < s_n\} = \{x : s_{n+1} \leq d(x, E_n) < s_n\} = F_n \backslash F_{n+1}.$$

Therefore, we can write w to be the square of some function of $d(x, E)$; in doing so we can suppose $f = 1$ if $r \geq r_0$, f is non-increasing, and $1 \leq \frac{f(r)}{f(2r)} \leq 1 + \tau$ where τ can be made arbitrarily small (because A_n can be made to grow arbitrarily slowly).

We finish by sketching the proof that there exists no quasiconformal map f on the plane such that $Jf \sim w$ on the plane (where J refers to the Jacobian). To do this it suffices to show that if f is a quasiconformal map on the plane with $Jf \sim w$ then $f(E)$ contains a rectifiable curve γ; as γ blows up on E, this would imply that f^{-1} has Jacobian vanishing on γ so $f^{-1}(\gamma)$ is a point which contradicts that f is a homeomorphism.

The main result we will use is the following 'good path' limit, whose proof (a geometric argument depending on various properties of quasiconformal maps) we omit for lack of time.

Lemma 12. *There exists $n_0 \in \boldsymbol{N}$ such that if $k \geq n_0$ and γ is a good path for $f(E_k)$ (i.e. a polygonal path contained in $f(E_k)$ whose vertices lie on the boundaries of the images of nth generation squares) then there is a good path γ' for $f(E_{k+1})$ such that each vertex of γ is also a vertex of γ' and $|\gamma'| \leq |\gamma|(1 + \frac{C_1}{M_n^2} + \frac{C_2}{N_n^3})$ where C_1, C_2, n depend only on the quasiconformal map f.*

Armed with this lemma, we obtain the rectifiable curve in $f(E)$ by a compactness argument; extending a line segment in $f(E_{n_0})$ gives us a good path for $f(E_{n_0})$ and iteratively applying the good path lemma gives us a sequence $\{\gamma_k\}$ of good paths in $f(E_k)$ of uniformly bounded length; parametrizing these paths appropriately and extracting a uniformly convergent subsequence yields (in limit) a good path contained in $\cap f(E_k) = f(E)$. This produces our desired contradiction so w is indeed comparable to no quasiconformal Jacobian.

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11 Removability theorems for Sobolev functions and quasiconformal maps

after P. W. Jones and S. K. Smirnov [4], and P. Koskela [3]
A summary written by Nicolae Tecu

Abstract

We give sufficient conditions sets to be Sobolev and quasiconformally removable.

11.1 Introduction

We start with presenting the main results in [4] on quasiconformal removability and continue with those in [3]. The paper is organized as follows: we will first introduce several necessary notions and make a few introductory remarks. We will continue with a presentation of the main theorems and, in the end, outline the proofs.

11.2 Definitions and introductory remarks

The setting of all the theorems is $\mathbb{R}^n$. Ω is an open set. In the following $W^{1,p}(\Omega)$ will denote the functions in $L^p(\Omega)$ whose distributional partial derivatives are also functions in L^p. This means that $u \in W^{1,p}(\Omega)$ if $u \in L^p(\Omega)$ and there are functions $\partial_j u \in L^p(\Omega), j = 1, \ldots, n$ such that

$$\int_\Omega u \partial_j \psi dx = -\int_\Omega \psi \partial_j u dx \tag{1}$$

for all test functions $\psi \in C_0^\infty(\Omega)$ and all $1 \le j \le n$.

Definition 1. *A compact set $K \subset U$ is quasiconformally removable inside domain U, if any homeomorphism of U which is quasiconformal on $U \setminus K$ is quasiconformal on U.*

Definition 2. *A compact set $K \subset U$ is called $W^{1,p}$-removable for continuous functions inside domain U, if any function, continuous in U and in $W^{1,p}(U \setminus K)$ belongs to $W^{1,p}(U)$.*

Definition 3. *A closed set $E \subset U$ is called $W^{1,p}$-removable or p-removable if $W^{1,p}(U) = W^{1,p}(U \setminus E)$.*

Note the difference between the last two definitions: in definition 2 the Sobolev function is assumed to be continuous in U, while in definition 3 this doesn't happen.

A few observations are in order. E is removable for $W^{1,p}$ if and only if for each $x \in E$ there is $r > 0$ such that $W^{1,p}(B(x,r) \setminus E) = W^{1,p}(B(x,r))$ as sets. Secondly, as smooth functions are dense in $W^{1,p}(\Omega \setminus E)$, it is sufficient to verify relation (1) for $u \in C_0^1(\Omega \setminus E) \cap W^{1,p}(\Omega \setminus E)$ and $\psi \in C_0^\infty(\Omega)$.

In the following K will be a compact set which is the boundary of a connected domain Ω. We may assume it is bounded. In this context we consider the Whitney decomposition of the domain: $\mathcal{W} = \{Q\}$. We will denote by $l(Q)$ the side length of cube Q and by $f(Q)$ the mean value of f in that cube. If Q is any cube, aQ denotes the concentric cube with the side length a times that of Q.

Definition 4. *Fix a family Γ of curves starting at the fixed point $z_0 \in \Omega$ and accumulating to $\partial\Omega$ such that their accumulation sets cover $\partial\Omega$. The "shadow" $SH(Q)$ of a cube Q is the closure in $\partial\Omega$ of the union of all curves $\gamma \in \Gamma$ starting at z_0 and passing through Q. Denote by $s(Q)$ the diameter of $SH(Q)$.*

Remark 5. *If δ is small enough, any curve in Γ will pass though at least one Whitney cube of that size. Thus $\partial\Omega$ is covered by shadows of finitely many cubes of sufficiently small size. This, in turn, implies that the Lebesgue volume of $\partial\Omega$ is zero.*

Definition 6. *Given a domain Ω we can consider the metric on it with the volume element $\frac{|dz|}{dist(z,\partial\Omega)}$. This will be denoted $dist_{qh}(.,.)$ and called quasi-hyperbolic metric.*

The quasihyperbolic metric is geodesic (for more information see [2]).

Definition 7. *We call a set $E \subset \mathbb{R}^{n-1}$ p-porous, $1 < p < n-1$, if for each H^{n-1}-a.e. $x \in E$, there is a sequence (r_i) and a constant C_x such that $r_i \to 0$ and each $(n-1)$-dimensional ball $B(x,r_i)$ contains a ball $B_i \subset B(x,r_i) \setminus E$ of radius no less than $C_x r_i^{(n-1)/(n-p)}$. For $n-1 \leq p < n$ we replace the balls B_i by continua F_i of diameters no less than $C_x r_i^{(n-1)/(n-p)}$. For $p = n$, E is p-porous if the diameter of F_i is no less than $C_x r_i exp(-1/C_x r_i)$.*

11.3 Main results

We now state the theorems.

Theorem 8. *If for some $p \geq 1$ a domain $\Omega \subset \mathbb{R}^n$ satisfies*

$$\sum_{Q \in \mathcal{W}} (s(Q)/l(Q))^{p'(n-1)} |Q| < \infty \tag{2}$$

where $1/p + 1/p' = 1$ then $K = \partial\Omega$ is $W^{1,p}$-removable for continuous functions.

In particular, for $p = n$ this gives the following

Corollary 9. *If Ω satisfies*

$$\sum_{Q \in \mathcal{W}} s(Q)^n < \infty \tag{3}$$

then $K = \partial\Omega$ is quasiconformally removable.

It is sufficient to include in the sum above only the cubes Q which are in a neighborhood of K and so there is no loss of generality if we restrict ourselves to bounded domains Ω.

If condition (3) holds, every curve in Γ starting at z_0 has exactly one landing point and each point on $\partial\Omega$ is a landing point of such a curve. If $z \in SH(Q)$ there is a such a curve passing through Q.

Theorem 10. *If for some fixed $z_0 \in \Omega$, where $\Omega \subset \mathbb{R}^n$ is a domain, satisfies*

$$dist_{qh}(., z_0) \in L^n(\Omega_K, m) \tag{4}$$

then $K = \partial\Omega$ is quasiconformally removable and $W^{1,n}$-removable for continuous functions.

Here Ω_K is a neighborhood of K in Ω(only integration near K is needed).

Theorem 11. *A totally disconnected closed set $E \subset \mathbb{R}^{n-1}$ is $W^{1,p}$-removable for $p > n$.*

Theorem 12. *If a set $E \subset \mathbb{R}^{n-1}$ is p-porous, $1 < p \leq n$, then E is removable for $W^{1,p}$ in $\mathbb{R}^n$. In addition, for each $1 < p \leq n$ there is a $p-porous$ set $E \subset \mathbb{R}^{n-1}$ that is not removable for $W^{1,q}$ for any $q < p$.*

11.4 Outline of proofs

Theorem 8 follows from the following proposition.

Proposition 13. *If Ω satisfies* (2)*, then any continuous function f, which belongs to $W^{1,p}$ for bounded subsets of K^C is $ACL(\mathbb{R}^n)$.*

Any continuous function $f \in W^{1,p}(K^C)$ is by the proposition in ACL($\mathbb{R}^n$). Since K has Lebesque measure 0 this implies $f \in W^{1,p}(\mathbb{R}^n)$(see [6], theorem 2.1.4). Any homeomorphism f which is quasiconformal in the complement of K, is also an element on $W^{1,n}_{loc}(K^C)$ and thus ACL($\mathbb{R}^n$) by the proposition. Since K has volume zero, $f \in$ACL($\mathbb{R}^n$) implies that f is quasiconformal in $\mathbb{R}^n$, which is what we wanted (see [5], Section 34).

Outline of proof of proposition 13. Consider a bounded domain U which contains K and let l be a line parallel to coordinate axis λ. The main part of the proof consists in proving that the total variation of f on $l \cap U$ equals the total variation of f on $l \cap U \setminus K$. If this holds then one can prove f is absolutely continuous on l.

f is in $W^{1,p}(U \setminus K)$ and this means that the partial derivatives of f exist almost everywhere and are in $L^p(U \setminus K)$. However, U is a bounded set and hence the partial derivatives are also in $L^1(U \setminus K)$. Fubini's theorem implies that $\partial_\lambda f \in L^1(l \cap U \setminus K)$ for almost every line l parallel to λ. Thus we see that the total variation of f on $l \cap U \setminus K$ is $\int_{l \cap U \setminus K} |\partial_\lambda f|$ for almost every line l in that particular direction.

Denote the total variation of f on $l \cap U$ by $\int_{l \cap U} |\partial_\lambda f|$. In order to prove that the total variations are the same, we will prove that

$$\int \int_U |\partial_\lambda f| = \int \int_{U \setminus K} |\partial_\lambda f|, \tag{5}$$

where the double integrals are to be understood as integrals over all lines parallel to λ of the total variations. Fubini's theorem gives us the desired result.

To prove (5) one starts by approximating $\int_{l \cap U} |\partial_\lambda f|$ by expressions of the type

$$\sum_j |f(x_j) - f(y_j)| + \int_{l \cap U \setminus \cup_j [x_j, y_j]} |\partial_\lambda f| \tag{6}$$

where $[x_j, y_j]$ cover $l \cap K$, with $x_j, y_j \in l \cap K$. Fix now one of these intervals. Since the shadows of the Whitney cubes of size Δ (picked at the end) cover

K we can find a partition of this intervals $\{u_i\}$ such that $u_0 = x_j, u_n = y_j$ and for any other i we either have $(u_i, u_{i+1}) \subset K^C$ or u_i, u_{i+1} belong to the same shadow $SH(Q_i)$. In the first case we have the estimate

$$|f(u_i) - f(u_{i+1})| \leq \int_{[u_i,u_{i+1}]} |\partial_\lambda f|. \tag{7}$$

In the second case we can find curves $\gamma_{i1}, \gamma_{i2} \in \Gamma$ which connect u_i and u_{i+1} to Q_i respectively. From these two one can obtain a curve which connects u_i to u_{i+1} and passes through Q_i - call it γ_i. With some care one then gets

$$|f(u_i) - f(u_{i+1})| \leq 2^n \sum_{Q\cap\gamma_i\neq\emptyset} |\partial_\lambda f|(Q)l(Q),$$

where the sum is taken over all Whitney cubes intersecting γ_i. All the Whitney cubes in the sum above are of size at most Δ and can be chosen to be in U.

Adding up everything and making a few more reductions one obtains

$$|f(x_j) - f(y_j)| \leq \int_{[x_j,y_j]\setminus K} |\partial_\lambda f| + 2^n \sum_{SH(Q)\cap[x_j,y_j]\neq\emptyset} |\partial_\lambda|(Q)l(Q).$$

This implies

$$\int_{l\cap U} |\partial_\lambda f| \leq 2^n \sum_{SH(Q)\cap l\neq\emptyset} |\partial_\lambda f|(Q)l(Q) + \int_{l\cap U\setminus K} |\partial_\lambda f|,$$

where the Whitney cubes appearing in the relation are of size at most Δ. By Fubini we get

$$\int\int_U |\partial_\lambda f| \leq 2^n \sum |\partial_\lambda f|(Q)l(Q)s(Q)^{n-1} + \int\int_{U\setminus K} |\partial_\lambda f|.$$

By Hoelder's inequality

$$\sum |\partial_\lambda f|(Q)l(Q)s(Q)^{n-1} \leq \left(\sum |\partial_\lambda f|(Q)^p|Q|\right)^{1/p} \left(\sum (s(Q)/l(Q))^{p'(n-1)}|Q|\right)^{1/p'}.$$

The cubes that appear in the sums are those of size less than Δ. While the first sum is finite by the fact that $f \in W^{1,p}(U \setminus K)$, condition (2) implies the second sum goes to zero as $\Delta \to 0$, which proves the proposition. □

Outline of proof of theorem 10. One considers the center of some Whitney cube $Q(z_0)$. Now join the centers of any two adjacent cubes by intervals and define the function q: $q(Q) =$ the number of intervals in the shortest chain joining the centers of $Q(z_0)$ and Q. One can remove redundant intervals such that $q(Q)$ is preserved and the collection of intervals is a tree. We then have $q(Q) \asymp \mathrm{dist}_{qh}(Q, z_0)$ for all $Q \in \mathcal{W}$. This transforms (4) into

$$\sum_{Q\in\mathcal{W}} q(Q)^n l(Q)^n \leq \infty \tag{8}$$

The next step consists in defining a collection of curves Γ: take all the chains of intervals in the tree above that contain an infinite number of intervals. One can prove that each point in K is a landing point of exactly one of these curves. Then, for each cube Q one can find a curve from Γ which has length comparable to $s(Q)$. Finally, one can prove that

$$\sum_{Q\in\mathcal{W}} s(Q)^n \leq C \sum_{Q\in\mathcal{W}} l(Q)^n q(Q)^n. \tag{9}$$

The proof is complete by applying Theorem 8 with $p = n$. □

Outline of proof of theorem 11. The theorem is proved first for $n = 2$ and $p > 2$. The set E is closed. Since removability is a local property, we may assume E is compact and further that it is a subset of $(0, 1)$. One can show that p-removability is equivalent to the H^1-a.e. equality of u^+ and u^- on E, where $u^+(x) = \lim_{0<t\to 0} u(x_1, t), x = (x_1, 0) \in W$, and $u^-(x)$ is defined analogously. Next, one can use the Sobolev embedding theorem to argue that u is uniformly Hoelder continuous both in the upper and lower halves of the ball $B(0, 2)$. Since E is totally disconnected (the only connected components are points), it has empty interior and a series of approximations gives that the difference of $u^+(x)$ and $u^-(x)$ can be made as small as one likes. For larger n one uses this base case and proceeds by induction using also integration by parts. □

Outline of proof of theorem 12. Consider first the case $n = 2$. We may make the same assumptions as in the previous proof. The key idea is to prove that $u^+(x) = u^-(x)$ for H^1-a.e. $x \in E$. One assumes the opposite and uses the fundamental theorem of calculus and Hoelder's inequality for $p <$

2 or replaces u by a harmonic function (as these minimize the "energy" $\int_{B(x,r)} |\nabla \cdot|^2 dx$) and uses capacity estimates for $p = 2$ to prove that

$$\int_{B(x,2r_i)} |\nabla u|^p dx \geq C_x r_i \tag{10}$$

for a sequence (r_i) which decreases to zero. This is a contradiction to the fact that

$$\lim_{z \to 0} \frac{1}{r} \int_{B(x,r)} |\nabla u|^p dx = 0. \tag{11}$$

for H^1-a.e. $x \in B(0,2)$(see [6], p.118).

Since the last fact is true for all n, one uses the same strategy for $n > 2$. One may assume $E \subset [0,1]^{n-1}$ and $u \in W^{1,p}(\Omega \setminus E)$. There exists a p-harmonic function v with the Dirichlet data given by u. Then $u - v \in W_0^{1,p}(\Omega \setminus E)$ and it suffices to show that E is removable for v. As v is p-harmonic and $\int_{\Omega \setminus E} |\nabla v|^p dx < \infty$, v has upper and lower non-tangential limits. As before, it is enough to show that

$$\int_{B(x,r_i)} |\nabla v|^p dx \geq C r_i^{n-1} \tag{12}$$

for $x \in E$ where the non-tangential limits do not coincide and where the porosity condition holds. This is accomplished by a series of reductions and capacity estimates.

Both for $n = 2$ and $n > 2$ the porosity provides sets of "considerable" size on which the function u or v take "well-separated" values. Then one uses averages or sequences of averages combined with the Sobolev-Poincare inequality(or Trudinger if $n = p$) to get a lower bound for the gradient. In the end one uses some covering arguments(see e.g. [1]). □

To finish theorem 12 we use the following result

Theorem 14. *Suppose that* $I^{n-1} \setminus E = \cup_{i=1}^{\infty} Q_i$, *where* $I = (0,1)$ *and each* Q_i *is an open cube in* I^{n-1}. *If* $1 < p < n$ *and*

$$\sum_{i=1}^{\infty} diam(Q_i)^{n-p} < \infty \text{ and } H^{n-1}(I^{n-1} \setminus \cup_{i=1}^{\infty} 2Q_i) > 0 \tag{13}$$

then E is not p-removable. If

$$\sum_{i=1}^{\infty} \log(1/diam(Q_i))^{1-n} < \infty \text{ and } H^{n-1}(I^{n-1} \setminus \cup_{i=1}^{\infty} diam(Q_i)^{-1/2} Q_i) > 0 \tag{14}$$

then E is not n-removable.

It is now sufficient to construct a p-porous compact $E \subset I^{n-1}$ such that E is not removable in $\mathbb{R}^n$ for $q < p$. We only consider the case $p < n$. The set is going to be a Sierpinski-type set. One starts by removing a cube Q_1 of side length $s2^{-A}$ (where $1/4 \leq s \leq 1/2$ and $A = (n-1)/(n-p)$) from the center of I^{n-1}. The remainder can be divided in two types of cubes. Some of them are going to be translates of Q_1 on the coordinate directions and the others are going to have side length $l_1 = (1 - s2^{-A})/2$ and are going to be 2^{n-1} of them. Denote the collection of all these cubes by $\mathcal{W}_1$. Next step consists in deleting a cube of side length $s2^{-2A}$ for an appropriate s from the center of *each* cube in $\mathcal{W}_1$ which has side length at least $l_1/2$. Subdivide the remainder (of each cube) as above and get a collection $\mathcal{W}_2$. Let l_2 be the largest side length of a cube in $\mathcal{W}_2$. Continue in this fashion with the deleting and subdividing process to get a sequence of collections $\mathcal{W}_i$. Set $E = \cap \mathcal{W}_i$. Then E will be p-porous and condition (13) will be satisfied for $1 < q < p$. The previous theorem shows that this E is not q-removable for all $1 < q < p$.

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12 Measurable Differentiable Structures and the Poincare Inequality

after Stephen Keith
A summary written by Armen Vagharshakyan

Abstract
The techniques developed in [2] are applied to improve the differentiable structure presented in [1]. It is shown that the coordinate functions of a differentiable structure can be taken to be distance functions. During the proof, the differential of a function contained in $H_{1,p}$ is described in terms of approximate limits.

12.1 Preliminary Notations

A metric measure space $< X, d, \mu >$ is a set X equipped with a metric d and a Borel measure μ on it. Note: the measures that we consider will be $\sigma-$finite. We say that the measure μ is doubling if there exists a constant $C > 0$ s.t. $\mu(B(x, 2r)) \leq C\mu(B(x, r))$ for any $x \in X$ and $r > 0$. Balls $B(x, r) = \{y : d(y, x) < r)$ are always assumed to have positive radius $0 < r < \infty$. For a function $f : X \to R$ define $LIP\, f = sup_{x \neq y} \frac{|f(y)-f(x)|}{d(y,x)}$ Then $LIP(X) = \{f : X \to R;\, LIP\, f < \infty\}$ is the space of Lipschitz functions. Let $LIP_0(X)$ be the space of Lipschitz functions with compact support. Also define $(lipf)(x) = liminf_{r\to 0} sup_{d(y,x)<r} \frac{|f(y)-f(x)|}{r}$ and $(Lipf)(x) = limsup_{r\to 0} sup_{d(y,x)<r} \frac{|f(y)-f(x)|}{r}$. Write $aplim_{y\to x} f(y) = A$ iff for any $\epsilon > 0$ we have $\lim_{r\to 0} \frac{\mu(B(x,r)\cap\{|f(y)-A|>\epsilon\})}{\mu(B(x,r))} = 0$.

We say that the metric measure space $< X, d, \mu >$ admits p-Poincare inequality (for some $p \geq 1$) iff

1. balls have positive measure: $0 < \mu(B(x, r)) < \infty$, for any $x \in X$ and $r > 0$

2. for any $f \in LIP(X)$ and any ball $B(x, r)$:

$$⨍_{B(x,r)} |f - f_{B(x,r)}| \leq Lr \left(⨍_{B(x,Lr)} lipf(x)^p d\mu \right)^{1/p}$$

12.2 Main Theorem

Cheeger (see [1]) demonstrated that metric measure spaces satisfying the Poincare inequality admit a "differentiable structure" with which Lipschitz functions can be differentiated almost everywhere. To be precise:

Theorem 1 (Cheeger)**.** *(see [1]) Let $< X, d, \mu >$ be a metric mesure space, which admits a p-Poincare inequality, with complete metric d and doubling measure μ then there exists a sequence $\{< X_\alpha, \vec{v}_\alpha >\}_\alpha$ (called "a strong measurable differentiable structure") where X_α's are measurable subsets of X, and v_α's are vector functions defined on X, so that*

1. $X = \bigcup X_\alpha$

2. $\vec{v}_\alpha = \{v_\alpha^1, \ldots v_\alpha^{N(\alpha)}\}$ where each of coordinate-functions v_α is in $LIP(X)$. and the number of coordinate functions is unformly bounded $\quad 1 \leq N(\alpha) \leq N < \infty$

3. For each $f \in LIP(X)$ and each X_α there exists a.e. unique (=unique up to a set of zero measure) function $\lambda_{f,\alpha} : X_\alpha \to R^{N(\alpha)}$ so that:

$$f(y) = f(x) + \vec{\lambda}_{f,\alpha}(x) \cdot [\vec{v}_\alpha(y) - \vec{v}_\alpha(x)] + o(d(y,x)), \quad a.e. \ x \in X_\alpha$$

The main result of this article is that we can take all v_α^i to be distance functions i.e.

Theorem 2 (Main Theorem)**.** *Let $< X, d, \mu >$ be a metric mesure space, which admits a p-Poincare inequality, with complete metric d and doubling measure μ then there exists a strong measurable differentiable structure $\{< X_\alpha, \vec{v}_\alpha >\}$ where all v_α^i are distance functiuons i.e.*

$$v_\alpha^i(x) = d(x, x_\alpha^i) \ \ for \ \ some \ \ x_\alpha^i \in X.$$

In order to prove the Main Theorem we need to establish some properties of Sobolev spaces over given metric spaces which admit a strong measurable differentiable structure. So let's introduce Sobolev spaces right now.

Sobolev Space. Consider the following norm on $LIP_0(X)$

$$||f||_{1,p} = ||f||_p + ||Lipf||_p$$

Denote the completion of $LIP_0(X)$ with respect to this norm by Sobolev space $H_{1,p}$.

Remark. It is essentially shown in [1] that the space $H_{1,p}$ is reflexive for $p > 1$.

Also denote

$D(X) = \{h(d(x,x_1), \ldots, d(x,x_n));\ h \in C^\infty(R^n, R); h \ has \ compact \ support \ ; n \in N\}$.
It turns out that $D(X)$ is everywhere dense in $H_{1,p}$ (see rroposition 4).

Cotangent Space T^*X_x. For $x \in X_\alpha$ consider

$$\{\vec{\lambda} \cdot (\vec{v}_\alpha(y) - \vec{v}_\alpha(x))\}_{\vec{\lambda} \in R^{N(\alpha)}}$$

This is an $N(\alpha)$ dimensional space of functions for a.e. $x \in X_\alpha$. Of course, the definition depends on the choice of α. If $x \in X_\alpha \bigcap X_\beta$ then we have 2 (generally speaking, different) linear spaces

$\{\vec{\lambda}_1 \cdot (\vec{v}_\alpha(y) - \vec{v}_\alpha(x))\}_{\vec{\lambda}_1}$ and $\{\vec{\lambda}_2 \cdot (\vec{v}_\beta(y) - \vec{v}_\beta(x))\}_{\vec{\lambda}_2}$

Let's identify $\vec{\lambda}_1 \cdot (\vec{v}_\alpha(y) - \vec{v}_\alpha(x))$ with $\vec{\lambda}_2 \cdot (\vec{v}_\beta(y) - \vec{v}_\beta(x))$ iff their difference is $o(d(y,x))$. This identification is a linear one-to-one correspondence for a.e. $x \in X_\alpha$. And T^*X_x will now be defined to be the linear space of functions with the precision of that identification.

Note, the differential $(df)(x) = \vec{\lambda}_{f,\alpha}(x) \cdot [\vec{v}_\alpha(y) - \vec{v}_\alpha(x)]$, $x \in X_\alpha$ is a well-defined element of T^*X_x.

S.Keith (see [2]) generalized Cheeger's theorem for any subset of $LIP(X)$. Applying this generalization for the set of distance functions, we get

Proposition 3. *There exists a natural number N_0 s.t.*

Every set $A \subset X$ with positive measure has a subset $W \subset A$ with positive measure so that W satisfies the following properties:

Property 1. There exists a vector-function $\vec{\rho}(x) = \{\rho_1(x), \dots \rho_N(x)\}$, $1 \le N \le N_0$ whose components are distance functions: $\rho_i(x) = d(x, x_i)$

s.t the functions $(d\rho_i)(x)$ are independent in T^*X_x for a.e. $x \in W$ and for any distance function ρ we have $(d\rho)(x) \in Span(\{(d\rho_i)(x)\}_i)$ for a.e. $x \in W$.

Property 2. $W \subset X_\alpha$ for some α.

Property 3. the functions $(d\rho_i)(x)$ span T^*X_x for a.e. $x \in W$, indeed it turns out that:

on the one hand, if $u \in D(X)$ then $(du)(x) \in Span(\{(d\rho_i)(x)\}_i)$ for a.e. $x \in X$ (see property 1 above and proposition 9) ,

on the other hand, for each function $v^i_\alpha(y) - v^i_\alpha(x)$ we can find a sequence $u_n \in D(X)$ s.t. $(du_n)(x) \longrightarrow v^i_\alpha(y) - v^i_\alpha(x)$ in T^*X_x for a.e. $x \in W$ (see consequence of proposition 4).

So, the system $\{(d\rho_i)(x)\}_i$ is a basis for the space T^*X_x. Hence, combining this fact with Cheeger's theorem, we get:

Property 4. For every function $f \in LIP(X)$ we have

$$f(y) = f(x) + \vec{\nu}_{f,\alpha}(x) \cdot [\vec{\rho}_\alpha(y) - \vec{\rho}_\alpha(x)] + o(d(y,x)), \ \ for \ \ a.e. \ \ x \in W$$

Property 5. The function $\vec{\nu}_{f,\alpha}(x)$ mentioned in Property 4 is a.e. unique, as: $\vec{\lambda}_{f,\alpha}(x)$ is a.e. unique on W and the functions $(d\rho_i)(x)$ constitute a basis for T^*X_x for a.e. $x \in W$.

Proof of Main Theorem. Define the property P_{N_0} to hold for $W \in X$ if the set W satisfies Properties 1 through 5. We know that every set A of positive measure has a subset W which satisfies P_{N_0}, hence the set X can be represented in a countable decomposition:

$$X = \bigcup_1^\infty W_n \bigcup Z,$$

where the set Z has zero measure whereas all sets W_n are disjoint, have positive measure, and satisfy P_{N_0} .

Let $\vec{\rho_i}$ be the vector-function of distance functions provided by *Property 1* for the set W_i then

$$\{< W_i, \vec{\rho_i} >\}_i$$

will be a strong measurable differentiable structure for $< X, d, \mu >$,

s.t. each ρ_i^j is a distance function.

12.3 Auxilary Results

We construct a canonical collection of norms on spaces T^*X_x, defined for almost every $x \in X$ in the following way:

Norm on the space T^*X_x is defined in the following way

$$\left|\vec{\lambda} \cdot (\vec{v}_\alpha(y) - \vec{v}_\alpha(x))\right|_x = Lip(\vec{\lambda} \cdot (\vec{v}_\alpha(y) - \vec{v}_\alpha(x)))(x)$$

It is well-defined on T^*X_x and it is a norm on T^*X_x for a.e. $x \in X_\alpha$.

Note: $f \in LIP(X)$ then $|(df)(x)|_x = (Lipf)(x)$.

Description of Sobolev Space . We can redefine the Sobolev space $H_{1,p}$ to be the completion of $LIP_0(X)$ with respect to

$$||f||_{1,p} = ||f||_p + |||(df)(x)|_x||_p$$

Now we can describe $H_{1,p}$. Elements of $H_{1,p}$ are pairs $< u, w >$ (where $u \in L_p$, $w(x) \in T^*X$, $|w(x)|_x \in L_p$) for whom exists a sequence $u_n \in LIP_0(X)$ s.t.

$||u_n - u||_p \to 0$ and $|||(du_n)(x) - w(x)|_x||_p \to 0$

Proposition 4. *$D(X)$ is dense in $H_{1,p}$.*

Proof of proposition 4.

I splitted the proof into two steps:

Step 1.

Suppose, we show that for any $u \in LIP_0(X)$ there exists a sequence $u_n \in D(X)$ s.t.

1. $u_n \to u$ in L_p and 2. $||u_n||_{1,p} < M < \infty$

then this will imply that $D(X)$ is dense in $H_{1,p}$.

Indeed, $H_{1,p}$ is reflexive (see [1]), so a ball in $H_{1,p}$ is weakly compact, so after passing to a subsequence, we can assume that $u_n \to < f, g >$ weakly in $H_{1,p}$ for some$< f, g > \in H_{1,p}$. In particular, we'll have $u_n \to < f, g >$ weakly in L_p. Hence $f(x) = u(x)$ for a.e. $x \in X$. This implies (see proposition 6) $g(x) = (du)(x)$ for a.e. $x \in X$, so $u_n \to u$ weakly in $H_{1,p}$. Therefore $LIP_0(X)$ is in weak closure of $D(X)$. Hence $D(X)$ is dense in $H_{1,p}$.

Step 2.

For a given function $u \in LIP_0(X)$ we want to construct the sequence $u_n \in D(X)$ which is demanded in step 1 of proposition 4. We will need a collection of well behaved balls, which is provided by the following lemma:

Lemma 5 (see [3]). *If μ is a doubling measure then for each $n \in N$ there exists a family of balls $B_i = B(x_i, 1/n)$ s.t. for every $k > 1$, every ball kB_i intersects with at most M_k balls kB_j, where M_k depends on the doubling constant of the measure μ and the number k only.*

Now consider the following functions

$$\phi_i(x) = \frac{\phi(nd(x, x_i))}{\sum_j \phi(nd(x, x_j))},$$

where $\phi : R_+ \to [0, 1]$ is any continuously differentiable function that satisfies

$\phi(x) = 1$ for $0 < x < 1$; and $\phi(x) = 0$ for $x > 2$.

Then the sequence

$$u_n(x) = \sum_i \phi_i(x) u_{B_i}$$

will be the desired one.

Consequence. First of all, for each $v^i_\alpha(x)$ one may construct a new function $c^i_\alpha(x)$ s.t. $c^i_\alpha \in LIP_0(X)$ and $(dv^i_\alpha)(x)$ and $(dc^i_\alpha)(x)$ will be the same element of T^*X_x when $x \in X_\alpha$.

Then, according to proposition 4, we can choose a sequence u_n in $D(X)$, s.t. $||u_n - c^i_\alpha||_{1,p} \to 0$. And, after passing to a subsequence, we'll have $|(du_n)(x) - (dv^i_\alpha)(x)|_x \to 0$ for a.e. $x \in X_\alpha$.

Proposition 6. *If* $< u, w > \in H_{1,p}(X)$ *then* w *is determined uniquely by* u.

The proof goes as follows:

Step 1. If $< u, w > \in H_{1,p}(X)$ then $\frac{1}{r} \fint_{B(x,r)} |u(y) - u(x) - w(x)(y)| \to 0$, as $r \to 0$ a.e. $x \in X$

Step 2. If $< u, w > \in H_{1,p}(X)$ then $aplim_{y \to x} \frac{|u(y)-u(x)-w(x)(y)|}{d(y,x)} \to 0$, a.e. $x \in X$.

Step 3. If $< u, w_1 > \in H_{1,p}(X)$ and $< u, w_2 > \in H_{1,p}(X)$ then $lim_{y \to x} \frac{|w_1(x)(y)-w_2(x)(y)|}{d(y,x)} = 0$, a.e. $x \in X$,
therefore $< u, w_1 >=< u, w_2 >$.

Proof of step 1. Denote $v(y) = u(y) - u(x) - w(x)(y)$ and $v_n(y) = u_n(y) - u(x) - w(x)(y)$ As we should prove the statement for almost every point $x \in X$, we are free to assume that x is a Lebesgue point for the function v. Then using that μ is doubling we may obtain that
$\fint_{B(x,r)} |v| d\mu \leq C \sum_{j=0}^{N} \fint_{B(x,r/2^j)} |v - v_{B(x,r/2^j)}| + o(1)$ as $N \to \infty$

So, a good upper estimate for an expression like $I = \fint_{B(x,s)} |v - v_{B(x,s)}|$ is needed. Note, we can't apply the Poincare inequality explicitly here as we might not have $v \in LIP(X)$, yet we can apply Poincare inequality for a function $v_n \in LIP(X)$, and note that $v_n \to v$ in L_p to get

$$I \leq Ls \left(\fint_{B(x,Ls)} |w(y) - d(w(x)(y))(y)|_y^p d\mu \right)^{1/p}$$

this last expression will have an estimate which will be admissible for us, as soon as we assume that $x \in X$ is a Lebesgue point for the function v and a density point for one of the sets X_α, and as it was noted above, we are allowed to make such assumptions.

For the other two steps, note that if the measure is doubling and p-Poincare inequality holds, then it can't be too much concentrated in the center of the ball, namely

Proposition 7. *If* μ *is doubling then there exists a constant* $0 < a < 1$ *s.t. for any* $x \in X, r < diamX$ *and* $k \in N$ *we have*

$$\mu(B(x, a^k r)) \leq (1 - a)^k \mu(B(x, r)).$$

and, in general, the measure can't be too much concentrated on a part of a ball. This is expressed quantitatively in the following way:

Proposition 8. *Exists an absolute constant $C > 0$ s.t. if $x \in B(x_0, r_0)$ and $r \leq r_0$ then*

$$\frac{\mu(B(x,r))}{\mu(B(x_0,r_0))} \geq \frac{1}{C}\left(\frac{r}{r_0}\right)^C$$

We use proposition 7 to prove step 2 of our proposition 6.

In order to show step 3 we prove that if $w \in LIP(X)$ then $aplim_{y \to x} \frac{w(y)}{d(y,x)} = 0$ implies $lim_{y \to x} \frac{w(y)}{d(y,x)} = 0$. The proof uses proposition 8.

Proposition 9. *Let $u \in D(X)$ then $(du)(x) \in Span(\{(d\rho)(x)\}_{\rho \text{ is a dist. function}})$ for a.e. $x \in W$.*

Indeed, if $u = h(\rho_1(x), \rho_2(x) \ldots \rho_n(x))$ where $\rho_i(x) = d(x, x_i)$ are some distance functions then one can prove

$$(du)(x) = \sum_{i=1}^{n} \frac{dh}{dx_i}(x) \cdot (d\rho_i)(x) \;\; for \;\; a.e. \;\; x \in X$$

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13 Sobolev met Poincaré

after Hajłasz and Koskela [3]
A summary written by Marshall Williams

Abstract

We summarize a number of results from the first half of [3], which generalizes the notion of Sobolev functions to the setting of metric spaces with a doubling measure. By taking the view that a function and its gradient should consist of a pair of functions satisfying a Poincaré inequality, the authors are able to extend a number of classical results to this more general setting.

13.1 Introduction

Notation

We mainly follow the notation of [3]. Throughout this summary, the term function is reserved exclusively for real valued functions. C, with or without a subscript, will always be a positive constant, as will $\sigma \geq 1$. $X = (X, d, \mu)$ will denote a metric measure space, with metric d and measure μ. $\Omega \subseteq X$ will always be an open subset of X. Unless otherwise stated, the measure μ is assumed to satisfy a doubling condition on the volume of balls,

$$\mu(2B) \leq C_d \mu(B), \tag{1}$$

where B is a ball, and λB is a concentric ball with λ times the radius of B.

When $A \subseteq X$ has positive measure, we set

$$\fint_A f \, d\mu = \mu(A)^{-1} \int_A f \, d\mu$$

Motivation

The main purpose of this paper is to generalize the theory of Sobolev spaces beyond the classical setting of $\mathbb{R}^n$. The authors develop this generalization in the setting of metric measure spaces, subject only to the doubling condition (1). This allows one to do analysis in very general settings, including topological manifolds, graphs, and fractals, as well as Carnot-Caratheodory

Spaces.

There are a number of different approaches to defining Sobolev functions in general metric spaces. An earlier approach by Hajłasz in [2] defined the spaces $M^{1,p}(X)$ to be the spaces of functions $u \in L^p(X)$ such that for some nonnegative $g \in L^p(X)$, the inequality

$$|u(x) - u(y)| \leq d(x,y)(g(x) + g(y)) \tag{2}$$

holds almost everywhere (i. e. for all $x, y \in X \backslash E$ with $\mu(E) = 0$.) Here μ need only be a Borel measure, not necessarily doubling.

Another approach is based on the notion of an upper gradient, introduced by Heinonen and Koskela in [5]. We say that a Borel function g is an upper gradient of a function u if the inequality

$$|u(x) - u(y)| \leq \int_\gamma g \, ds \tag{3}$$

is satisfied for every rectifiable curve γ joining any two points x and y.

In fact, upper gradients have been used by Cheeger [1] and Shanmugalingam [6] to define Sobolev spaces in a general setting. However, one downside is that the theory becomes trivial in spaces with few rectifiable curves, such as graphs or certain fractals. To get a theory with applications to these latter types of spaces, the authors take the perspective that a Sobolev function and its gradient ought to be related by a Poincaré inequality.

We recall that the classical Poincaré inequality in $\mathbb{R}^n$ states that for Lipschitz function $u : \mathbb{R}^n \to \mathbb{R}$ we have:

$$\left(\fint_B |u - u_B|^p \, dx \right)^{1/p} \leq C(n,p) r \left(\fint_B |\nabla u|^p \, dx \right)^{1/p}. \tag{4}$$

The idea in this paper will be to use a generalization of (4) to extend the notion of a Sobolev function to a general metric space. We will expand on this below, and will see that there is a close connection with Hajłasz's earlier approach in [2].

13.2 Some preliminaries

First, motivated by (4), we make the following definition:

Definition 1. *Let $\Omega \subseteq X$ be open. A pair of measurable functions $u \in L^1_{loc}(\Omega)$, $g \geq 0$ on Ω is said to satisfy a p–**Poincaré inequality** in Ω (or simply a p–Poincaré inequality, if $\Omega = X$) if for every ball B with $\sigma B \subseteq \Omega$ we have*

$$\fint_B |u - u_B| \, d\mu \leq C_P r \left(\fint_{\sigma B} g^p \, d\mu \right)^{1/p} \tag{5}$$

where $C_P > 0$, and $\sigma \geq 1$ are fixed constants. We will sometimes add "with data (C_P, σ)" to be specific.

We will also need to make use of the following truncation property, which generalizes a property of gradients in $\mathbb{R}^n$:

Definition 2. *A pair of functions u, g satisfying a p–Poincaré inequality is said to also satisfy the **truncation property** if for every $b \in \mathbb{R}$, $0 < t_1 < t_2 < \infty$, and $\epsilon = \pm 1$, the pairs $(\epsilon(u-b))_{t_1}^{t_2}$,g satisfy the same p–Poincaré inequality (i.e. with the same constants C_p and σ). Here $v_{t_1}^{t_2} = \min\{\max\{0, v - t_1\}, t_2 - t_1\}$.*

One of the main questions explored in [3] is to what extent a p–Poincaré inequality implies the following Sobolev inequality:

Definition 3. *We say a pair u,g satisfies a **global Sobolev inequality** for Ω (with data (p, q, C)) if*

$$\inf_{c \in \mathbb{R}} \left(\int_\Omega |u - c|^q \, d\mu \right)^{1/q} \leq C \left(\int_\Omega g^p \, d\mu \right)^{1/p}. \tag{6}$$

*We also say the pair satisfies a **weak Sobolev inequality** (with data (q, p, C, σ)) if for all balls B of radius r*

$$\inf_{c \in \mathbb{R}} \left(\fint_B |u - c|^q \, d\mu \right)^{1/q} \leq Cr \left(\fint_{\sigma B} g^p \, d\mu \right)^{1/p} \tag{7}$$

We will want to investigate the relationship between the Sobolev inequalities in 3 and the following weaker inequalities:

Definition 4. *We say that a pair u, g satisfies a* ***global Marcinkiewicz-Sobolev inequality*** *on Ω (with data (q, p, C) if*

$$\inf_{c \in \mathbb{R}} \sup_{t \geq 0} \mu(\{x \in \Omega : |u(x) - c| > t\}) t^q \leq C \left(\int_\Omega g^p \, d\mu \right)^{q/p}. \tag{8}$$

We also say the pair satisfies a ***weak Marcinkiewicz-Sobolev inequality*** *(with data (p, q, C, σ) if for each ball B of radius r,*

$$\inf_{c \in \mathbb{R}} \sup_{t \geq 0} \frac{\mu(\{x \in B : |u(x) - c| > t\}) t^q}{\mu(B)} \leq C r^q \left(\fint_{\sigma B} g^p \, d\mu \right)^{q/p}. \tag{9}$$

Finally, we recall a few miscellaneous definitions and theorems which will come in handy.

Definition 5 (Marcinkiewicz spaces)**.** *A function f is in the* ***Marcinkiewicz space*** *$L^p_w(X)$ if there is some $m > 0$ such that for all $t > 0$,*

$$\mu(\{|u| > t\}) \leq m t^{-p}.$$

Definition 6. *For $f \in L^1_l oc(\Omega)$ and $R > 0$, we define the* ***restricted (Hardy–Littlewood) maximal function*** *of f (on Ω),*

$$M_{\Omega,R} f(x) = \sup_{0 < r \leq R} \frac{1}{\mu(B(x,r))} \int_{\Omega \cap B(x,r)} |f| \, d\mu. \tag{10}$$

If $R = \infty$, we write $M_\Omega f(X)$, which we refer to as simply the ***maximal function*** *of f on Ω. We omit the subscript Ω in the case that $\Omega = X$.*

We will also need a version of maximal theorem of Hardy, Littlewood, and Wiener, adapted for doubling metric spaces. For a proof, see [4], chapter 2.

Theorem 7 (Maximal Theorem (14.13 in [3]))**.** *Let μ be doubling. Then for $t > 0$ we have*

$$\mu(\{x \in \Omega : M_\Omega u(x) > t\}) \leq C t^{-1} \int_\Omega |u| \, d\mu. \tag{11}$$

Thus M_Ω maps $L^1(\Omega)$ continuously into $L^1_w(\Omega)$. Also, for $1 < p \leq \infty$,

$$||M_\Omega u||_{L^p(\Omega)} \leq C ||u||_{L^p(\Omega)}. \tag{12}$$

C depends only on the doubling constant C_d and the exponent p.

We will need an elementary fact about doubling measures, namely that for some C_b, s depending only on the doubling constant C_d,

$$\frac{\mu(B)}{\mu(B_0} \geq C_b(r/r_0)^s \tag{13}$$

13.3 Summary of results

We prove a number of results for pairs u, g satisfying the Poincaré inequality (5).

First, we show that if we assume our pair also has the truncation property, then a Marcinkiewicz–Sobolev inequality implies a Sobolev inequality. That is,

Theorem 8 (2.1 and 2.3 in [3]). *Suppose $\mu(\Omega) < \infty$. Suppose also that every pair u,g satisfying a p–Poincaré inequality in Ω with data (C_P, σ) also satisfies the global Marcinkiewicz–Sobolev inequality (8), with data (q, p, C_1). Then any such pair that also has the truncation property (2) will in fact satisfy the global Sobolev inequality (6) with data (q, p, C_2), where $C_2 = 8 \cdot (4C_1)^{1/q}$.*

Similarly, if every pair u, g satisfying the p–Poincaré inequality with data (C_P, σ) satisfies (9) with data (q, p, C_1, σ) then every such pair with the truncation property (2) will in fact satisfy (7) with data (q, p, C_2, σ).

Next, we examine the relationship between pairs satisfying a p–Poincaré inequality and the spaces $M^{1,p}(X)$ (see the Introduction). From now on we assume μ is doubling. We prove:

Theorem 9 (3.1 in [3]). *Let $1 < p < \infty$. Then a function u is in $M^{1,p}(X)$ if and only if $u \in L^p(X)$ and there is a $g \in L^p(X)$ such that the pair u, g satisfies a q Poincaré inequality for some $0 < q < p$.*

We investigate spaces where pairs u, g with g an upper gradient of u, satisfy a p–Poincaré inequality. We look at an example:

Example 10 (4.2 in [3]). *The cone $X = \{(x_1, ..., x_n) \in \mathbb{R}^n : x_1^2 + ... + x_{n-1}^2 \leq x_n^2\}$ (equipped with Euclidean metric and Lebesque measure) has the property that every continuous u with and upper gradient g satisfy a p–Poincaré inequality, if and only if $p > n$.*

We also show a geometric consequence of a Poincaré inequality for upper gradients:

Theorem 11 (4.4 in [3])**.** *Suppose X is proper and path connected (and doubling), and $p \geq 1$. If each pair of a continuous function and its upper gradient satisfies a p-Poincaré inequality (with fixed data), then X is quasiconvex (i.e. every two points x and y can be joined by a path of length at most $Cd(x, y)$.)*

Finally, we prove a kind of Sobolev embedding theorem. We have:

Theorem 12 (5.1 in [3])**.** *Suppose the pair u,g satisfies a p–Poincaré inequality, and suppose further that s is as in (13). Let $p^* = sp/(s-p)$ when $p < s$, and similarly $q^* = sq/(s-q)$. Then the following hold for all balls B of radius r:*

- *If $p < s$, then*

$$\frac{\mu(\{x \in B : |u(x) - u_B| > t\})t^{p^*}}{\mu(B)} \leq Cr^{p^*} \left(⨍_{5\sigma B} g^p \, d\mu \right)^{1/p}$$

  For $p < q < s$ we have

$$\left(⨍_B |u - u_B|^{q*} \, d\mu \right)^{1/q^*} \leq Cr \left(⨍_{5\sigma B} g^q \, d\mu \right)^{1/q}$$

  If the pair has the truncation property 2 as well, then

$$\left(⨍_B |u - u_B|^{p*} \, d\mu \right)^{1/p^*} \leq Cr \left(⨍_{5\sigma B} g^p \, d\mu \right)^{1/p}$$

- *If $p = s$, then*

$$⨍_B \exp \left(\frac{C_1 \mu(B)^{1/s} |u - u_B|}{r ||g||_{L^s(5\sigma B)}} \right) \, d\mu \leq C_2$$

- *If $p > s$, then u is almost everywhere equal to a Hölder continuous function, and*

$$\sup_{x \in B} |u(x) - u_B| \leq Cr \left(dashint_{5\sigma B} g^p \, d\mu \right)^{1/p}$$

  For $x, y \in B_0$, we have

$$|u(x) - u(y)| \leq Cr_0^{s/p} d(x, y)^{1-s/p} \left(⨍_{5\sigma B_0} g^p \, d\mu \right)^{1/p}$$

All constants depend only on the data p, q, s, C_d, σ, C_P, and C_b.

13.4 Outline of proofs

We very briefly outline the proofs for some of the results above.

Proof of 8. Here the idea is to pick b exactly so that $u - b$ is nonnegative on at least half of the mass of Ω, and also nonpositive on half the mass. We estimate the positive and negative parts of $u - b$ separately, labeling each one v. The integral $\int_\Omega v^q \, d\mu$ can be broken into pieces where the integrand is controlled by powers of 2. The truncation property says that these pieces and the corresponding restrictions of the gradient still satisfy a p–Poincaré inequality, so by hypothesis we can use the Marcinkiewicz inequality (8) on each piece. Summing gives the desired estimate. The same idea works for the weak inequalities, where we restrict to balls. □

Proof of 9. $\Rightarrow$ This direction is simple. Integrating (2) over a ball B with respect to x and then y gives the desired result.
$\Leftarrow$ To see this we need to show that if u, g satisfy a p–Poincaré inequality, then we have

$$|u(x) - u(y)| \leq C d(x,y) \left((M_{2\sigma d(x,y)} g^p(x))^{1/p} + (M_{2\sigma d(x,y)} g^p(y))^{1/p} \right) \qquad (14)$$

for almost every $x, y \in X$. (This is 3.2 in [3]) Indeed, if we show this, then assuming g satisfies a q–Poincaré inequality, replacing p with q in the above result and applying the maximal theorem 7 to $g^q \in L^{p/q}$ gives the desired result.
All we have to do is show (14) holds almost everywhere. To do this, we let x be a Lebesque point of u and estimate $|u(x) - u_{B(x,d(x,y))}|$ with a series of successively smaller balls, using the doubling property to transform each term into an average that looks like the left hand side of a Poincaré inequality. Applying Poincaré and summing will give us our estimate. (We must estimate $|u_{B(x,d(x,y))} - u_{B(y,d(x,y))}|$ as well, but this is easily done using the doubling property and Poincar e.) This completes the proof. □

We should point out that a converse to the fact that Poincaré implies (14) is true as well. That is, if $u \in L^1_{loc}(X)$, $0 \leq g \in L^p_{loc}(X)$, and $1 < p < \infty$, then if (14) holds for almost all $x, y \in X$, then the pair u, g satisfies a p–Poincaré inequality, with data depending only on the constants in (14) (This is 3.3 in [3]). The way to see this is to average (14) over x and y in a ball B with radius r, then apply Cavalieri's principle and, after truncating the resulting integral, apply the maximal theorem 7 to $g^p \chi_{B'} \in L^1$, where B' is a slightly larger ball. Truncating the integral judiciously gives us the desired estimate.

Verification of example 10. To see that the cone in 10 satisfies a Poincaré inequality if and only if $p > n$, we first note that $log|log|x|| \in W^{1,n}(B(0, 1/2)$ and can be truncated to construct functions whose gradients have arbitrarily small L^n norm, yet which are constantly 1 on the lower cone, and constantly 0 on most of the upper cone, and thus have mean oscillation bounded away from 0, prohibiting an n–Poincaré inequality.

To show 5 holds for $p > n$, the trick is to use the classical Sobolev embedding theorem (see, e.g. [4] chapter 3) to get a pointwise estimate as in (14) for $|u(x) - u(y)|$, and then invoke the remark above that such an estimate implies a Poincaré inequality.

□

Proof of 11. The details of this argument are a bit technical, but the basic idea is fairly simple. For any pair of points $x, y \in X$, we need to construct a path from x to y of length at most $Cd(x, y)$. For each $k \in \mathbb{N}$, we construct a path γ_k from x to y. We then define, for each k, a kind of "pseudo–length" function $l_k(\gamma)$, and pick our paths γ_k so that $l_k(\gamma_k)$ is minimized (up to a constant). We define our pseudo length in such a way that $g \equiv 1$ is an upper gradient of each function $u_k(z) = \inf l_k(\gamma)$ where the infimum is taken over curves γ joining x and z. This lets us use (5) to estimate $|u_k(y)| \leq Cd(x, y)$. We thus have a sequence of curves with bounded pseudo length. Because of the way we define this pseudo length, we are able to modify our space a little bit to get a family of curves with bounded length. Fortunately, after we use a compactness argument to pass to a limiting curve, we can show that this limiting curve lies in our original, unmodified space X. □

Proof of 12. This is also quite technical, and in the interests of space, we won't go into detail here. The main idea, however, is to split it into two steps. First, we estimate $|u - u_B| \leq CJ^{\sigma,B}_{1,p}g$, where the right hand side is a kind of generalized Riesz potential. (This is 5.2 in [3]). We then use a fractional integration theorem (5.3 in [3]) to control $J^{\sigma,B}_{1,p}g$. Roughly speaking, we need to show that $J^{\sigma,B}_{1,p}$ maps L^p into $L^{p^*}_w$ (continuously) when $p < s$, and maps L^q into L^{q^*} for $p < q < s$. Also, for $p = s$, we need to show $J^{\sigma,B}_{1,p}g$ satisfies an exponential inequality similar to the one in the statement of the theorem for that case, and for $p > s$, we must show $J_{1,p}^{\sigma,B}$ maps L^p into L^∞. □

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