The Hofer-Zehnder Capacity of Symplectic manifolds

Presented by Olga Radko (UCLA)

8th September 2004

– Typeset by FoilTEX –

Hamiltonian mechanics

Phase space: $\mathbb{R}^{2n}$, $z = (\underbrace{q_1, \ldots, q_n}_{\text{cordinates}}, \underbrace{p_1, \ldots, p_n}_{\text{momenta}})$.

Hamiltonian function: $H(q, p, t)$.

Smooth path: $z(t) = (q(t), p(t)) : [0, 1] \to \mathbb{R}^{2n}$

Action form: $\lambda_H(z, t) = \sum_{i=1}^n p_i dq_i - H dt$

VARIATIONAL PRINCIPLE: $z(t)$ satisfies the Hamiltonian equations

$$\dot{z} = X_H(z)$$

iff $z(t)$ is a critical pt of the action functional

$$\Phi_H(z(t)) = \int_{z(t)} \lambda_H(z(t), t).$$

Here $X_H = -J\nabla H$ is the Hamiltonian vector field of H,

$\nabla H = (\partial H/\partial q,\ \partial H/\partial p)$ and $J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$.

Symplectic manifolds

STANDARD SYMPL. FORM: $\omega_0 = \sum_{i=1}^{n} dq_i \wedge dp_i$.

Vector bundle isomorphism $\tilde{\omega}_0 : TM \to T^*M$:

$$(\tilde{\omega}_0(X))(Y) = \omega_0(X, Y).$$

X_H defined by $X_H = \tilde{\omega}_0(dH)$.

SYMPLECTIC MANIFOLD (M^{2n}, ω):

1. ω is closed: $d\omega = 0$.

2. ω is non-deg. : $\tilde{\omega}$ is an isomorphism.

SYMPLECTOMORPHISM: $\varphi \in \mathrm{Diff}(M)$, $\varphi^*\omega = \omega$.

LOCAL STRUCTURE (Darboux): Around a pt x,

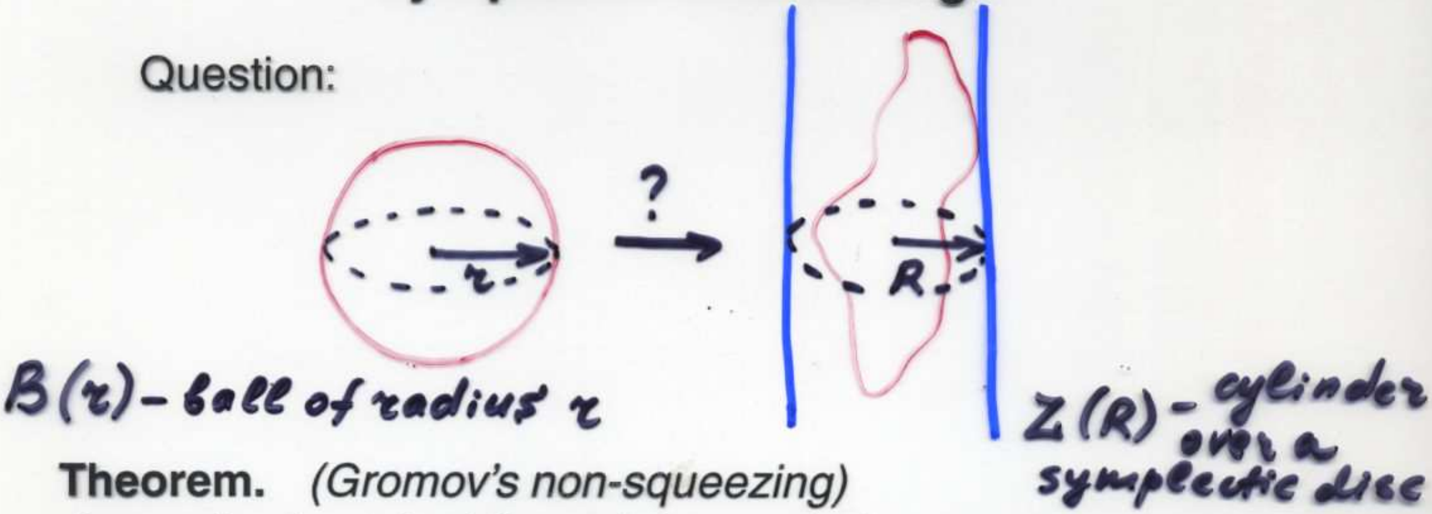
(M, ω), $\dim(M) = 2n$, "looks like" $(\mathbb{R}^{2n}, \omega_0)$.

VOLUME FORM: $\Omega = \frac{1}{n!}\omega \wedge \ldots \wedge \omega$;

TOTAL VOLUME $\mathrm{vol}(M, \omega) = \int_M \Omega$ – symplectic invariant.

Symplectic embeddings

Question:



Theorem. *(Gromov's non-squeezing)*
A symplectic embedding $B(r) \to Z(R)$ *exists iff* $z \le R$.

REMARKS:

1. Necessary condition: $\mathrm{vol}(B(r)) \le \mathrm{vol}(Z(R))$ (holds $\forall r, R$).
2. $\exists$ (Linear) vol.-preserving embedding $\psi : B(r) \to Z(R)$:

$$\psi(x, y) = (\varepsilon x_1, \frac{x_2}{\varepsilon}, x_3, \ldots, x_n; \varepsilon y_1, \frac{y_2}{\varepsilon}, y_3, \ldots, y_n), \quad \varepsilon < \frac{r}{R}.$$

However, ψ is symplectic iff $\varepsilon = 1$, which implies $r \le R$.

Thus, invariants other then total volume are needed.

Gromov's widths (a symplectic capacity):

$$w_G(M, \omega) = \sup\{\pi r^2 \ : \ B(r) \text{ embeds symplectically into } M\}.$$

Capacities

Let $\mathcal{M}^{2n} = \{(M, \omega) \ : \ \dim(M) = 2n\}$.

DEF. *Capacity* $c : \mathcal{M}^{2n} \to \mathbb{R}^{\geq 0} \cup \{+\infty\}$ satisfies

A1 (Monotonicity) If $\varphi : (M, \omega) \to (N, \tau)$ is symplectic,

then $c(M < \omega) \leq c(N, \tau)$.

A2 (Conformality) $c(M, \alpha\omega) = |\alpha| c(M, \omega)$.

A3 (Normalization) $c(B(1)) = \pi = c(Z(1))$.

PROPERTIES AND REMARKS:

1. c is a symplectic invariant.
2. $\exists\ c \Leftrightarrow$ Gromov's squeezing thm.
3. If c exists, might not be unique.
4. If $\dim = 2$, c is the total volume.
5. If $\dim \neq 2$, total volume is not a capacity.

QUESTION: Does a capacity exist?

Capacities Exist!

BUT THIS IS HARD TO PROVE! (see lecture 2)

Linear case

Mnfld	$\Longrightarrow$	Ellipsoid (image of $B(1)$ under linear map)
Capacity	$\Longrightarrow$	Linear capacity (only linear maps are allowed)

Lemma. *E can be put into the standard form $E_0 = \varphi(E) = \sum_{i=1}^{n} \frac{|z_i|^2}{r_i^2}$ by a symplectic map.*

$r = (r_1, \ldots, r_n)$ is the *spectrum* of E, $0 < r_1 \leq \ldots \leq r_n$.

Let E, E' be two ellipsoids with spectra r and r'.

Theorem. *Linear symplectic embedding $\varphi : E \to E'$ exists iff $r \leq r'$.*

Linear width (a linear capacity): $w_L(E) = \pi r_1^2(E)$.
Since $B(r_1) \subset \varphi(E) \subset Z(r_1)$, for any symplectic capacity

$$c(E) = \pi r_1^2 = w_L(E).$$

Dynamical interpretation of Linear capacities

Let $E = E(q) = \{q(z) < 1\}$ be an ellipsoid.

$\partial E = \{q(z) = 1\}$ is a compact hypersurface.

Distinguished periodic orbits of X_q on ∂E:

$w_i(t) = (0, \dots, z_i(t), \dots, 0)$,
where $z_i(t) = e^{\frac{2}{r_i^2}tJ} z_i(0)$, $|z_i(0)| = r_i$.

Action on distinguished orbits:

$$A(w_i(t)) = \frac{1}{2}\int_0^T \langle -J\dot{w}_i, w_i \rangle = \pi r_i^2.$$

Lemma. *For all $i = 1, \dots, n$:*

$$\pi r_i^2 = A(w_i(t)).$$

In particular,

$$c(E) = \pi r_1^2 = \min\{|A(z)| \ : \ z \text{ is a periodic orbit on } \partial E\}.$$

Symplectomorphism $\Leftrightarrow$ preserves capacities (Linear case)

Lemma 1. *Linear orientation-preserving map $\varphi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is symplectic iff it preserves the capacities of all ellipsoids.*

Proof. " $\Leftarrow$ ": If φ is not symplectic, $\exists\, v, u$ such that

$$0 < \lambda^2 = \omega(\varphi(v), \varphi(u)) < \omega(v, u) = 1.$$

Consider two bases:

$$\begin{aligned} \beta &= \{v, u, v_2, u_2, \dots, v_n, u_n\}, \\ \beta' &= \left\{ \frac{\varphi(v)}{\lambda}, \frac{\varphi(u)}{\lambda}, v_2', u_2', \dots, v_n', u_n' \right\}. \end{aligned}$$

Let $\gamma = \{e_1, f_1, \dots, e_n, f_n\}$ be the standard basis.

Let $\psi, \psi' \in \mathrm{Sp}(\mathbb{R}^{2n})$ be such that $\psi(\beta) = \gamma = \psi'(\beta')$.

Let $A = \psi'^{-1}\varphi\psi$. Then $A(e_1) = \lambda e_1, Af_1 = \lambda f_1$. This implies $A^T(B(1)) \subset Z(\lambda)$.

On the other hand, A^T preserves capacities.

For $\lambda < 1$, this gives a contradiction. □

Symplectomorphism $\Leftrightarrow$ Preserves capacities

Lemma 2. *The capacity of convex sets is continuous with respect to Hausdorff metric:*

$$d(U,V) = \max_{x\in U}(\min ||x-y||) + \max_{y\in V}(\min_{x\in U} ||x-y||).$$

Proof. If $d(U,V) < \delta$, then $\exists \varepsilon > 0$ s.t.

$$(1-\varepsilon)\cdot U \subset V \subset (1+\varepsilon)U.$$

Apply A1; then take the limit. □

Theorem. *(Ekeland-Hofer) A diffeomorphism φ is symplectic iff it preserves the capacities of all open sets.*

Proof. Consider the maps

$$\varphi_t(z) = \frac{\varphi(tz)}{t}.$$

$\varphi_t \in \mathrm{Diff}^+(\mathbb{R}^{2n})$ and φ_t preserves capacity of ellipsoids.

Let $\Phi = \lim_{t\to 0}\varphi_t = d\varphi(0)$(uniform on compact sets).

Hence, Φ is linear and preserves capacities of ellipsoids.

Thus, Φ is symplectic, and φ is symplectic at 0.

Similar argument works for $\Phi = d\varphi(z)\ \forall z \in \mathbb{R}^{2n}$. □

Rigidity

Corollary. *Symp*(M) *is* C^0*-closed in Diff*(M).
In other words, if

- $\varphi_j \to \varphi$ *uniformly on compact sets;*
- φ_j *are symplectic;*

Then φ *is symplectic.*

Proof. Let $\lim_{j\to\infty}\varphi_j = \varphi$ (uniform convergence).
Let $\varphi_j \in \mathrm{Symp}(M)$. Since c is continuous w.r.to Hausdorff metric, φ preserves c, and, thus is symplectic. □

This is analogous to

Theorem. *If* $f_n \to f$ *uniformly on compact sets, and* $f_n : \mathbb{C}^n \to \mathbb{R}$ *is analytic, then* f *is analytic.*

Proof. The integral

$$\int_\gamma f(z)dz$$

"behaves well" w.r.to uniform limits on compact sets. □

Symplectic Homeomorphisms

Q: How to define a symplectic homeomorphism?

VOLUME-PRESERVING CASE:

Homeomorphisms preserving the measure.

SYMPLECTIC CASE:

A homeomorphism φ is *symplectic* if φ preserves c.

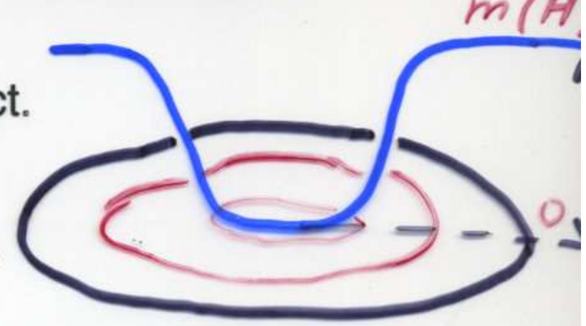
Thus, each c defines a subgroup $\mathrm{Hom}_c(M)$.

$\mathrm{Hom}_c(M)$ is C^0-closed in $\mathrm{Hom}(M)$.

If $\varphi \in \mathrm{Hom}_c(M)$ is smooth, it is in $\mathrm{Symp}(M)$.

The Hofer-Zehnder capacity

Consider Hamiltonians $H \in \mathcal{H}(M,\omega)$ such that



1. $H(M \setminus K) = m(H)$, K is compact.
2. $H(U) = 0$ for some open $U \subset K$.
3. $0 \le H(x) \le m(H)$ for all $x \in M$.

Here $m(H) = \max(H) - \min(H)$ is oscillation of H.

Definition. *Admissible Hamiltonians $\mathcal{H}_a(M) \subset \mathcal{H}(M)$: a periodic solution of $\dot{z} = H(z)$ is either constant or $T > 1$.*

Note: $\forall H \in \mathcal{H}_a(M)$ $\exists \varepsilon > 0$ such that $\varepsilon \cdot H \in \mathcal{H}_a(M)$.

Definition. *The Hofer-Zehnder capacity is*

$$c_0(M,\omega) = \sup\{m(H) \mid H \in \mathcal{H}_a(M,\omega)\},$$

the maximal possible oscillation of admissible functions on M.

If $c_0(M)$ is finite, then $\forall$ $H \in \mathcal{H}_a(M)$ with $m(H) > c_0(M)$, X_H has a non-constant periodic orbit which is "fast" ($T < 1$). Moreover, $c_0(M)$ if the infimum of numbers with this property.

Monotonicity and Conformality of c_0.

Lemma. c_0 *satisfies* $A1$.

Proof. $\varphi : (M_1, \omega_1) \to (M_2, \omega_2)$ – symplectic embedding.

For $H_1 \in \mathcal{H}_a(M_1)$ define

$$H_2(y) = \begin{cases} H_1(\varphi^{-1}(y)), & y \in \varphi(M_1); \\ 0, & y \in M_2 \setminus \varphi(M_1); \end{cases}$$

Since $m(H_2) = m(H_1)$, and φ intertwines X_{H_1} and X_{H_2}, the periodic orbits correspond to periodic orbits.
Thus $c_0(M) \le c_0(N)$. □

Lemma. c_0 *satisfies A2.*

Proof. Let $\psi : \mathcal{H}(M, \omega) \to \mathcal{H}(M, \alpha\omega)$ be the bijection

$$\psi(H) \equiv H_\alpha \doteq |\alpha| \cdot H.$$

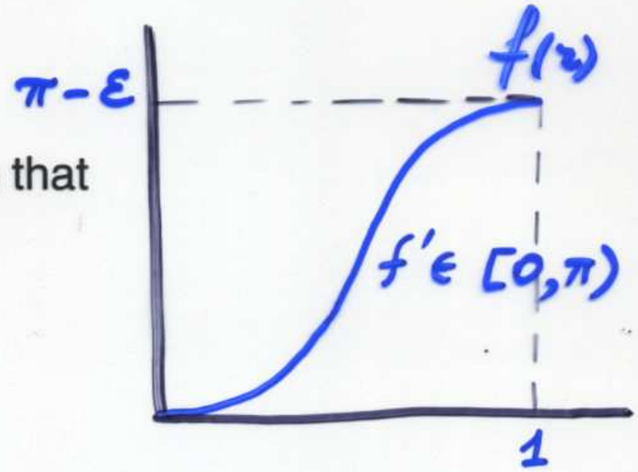
Since

$$(\alpha\omega)(X_{H_\alpha}, \cdot) = -dH_\alpha = -|\alpha| \cdot dH = |\alpha| \cdot \omega(X_H, \cdot).$$

By non-degeneracy, $X_{H_\alpha} = \pm X_H$. The periods are equal. □

Normalization (easy part)

Lemma. $c_0(B(1)) \geq \pi$.



Proof. Let $f : [0,1] \to \mathbb{R}$ be such that

- $f(r) = \pi - \varepsilon$ for r near 1;
- $f(r) = 0$ for r near 0;
- $f'(r) \in [0, \pi)$.

Let $H(z) = f(|z|^2)$. Then $H \in \mathcal{H}(M)$ and $m(H) = \pi - \varepsilon$.

The solutions of $\dot{z} = X_H(z)$ is $z(t) = e^{aJt}z(0)$,

where $a = 2f'(z(0)) \in [0, 2\pi)$.

If $a = 0$, $z(t) = \text{const}$.

If $a > 0$, $z(t)$ is periodic with $T = \frac{2\pi}{a} > 1$.

Thus, $H \in \mathcal{H}_a(M, \omega)$. Thus, $c_0(M) \geq m(H) = \pi - \varepsilon$.

Taking the limit $\varepsilon \to 0$, we obtain $c_0(M) \geq \pi$. □

Since ∃ a symplectic embedding $B(1) \to Z(1)$,

it follows that $c_0(Z(1)) \geq c_0(B(1))$.

Normalization (difficult part)

To prove that c_0 satisfies A3, it remains to show that

$$c_0(Z(1)) \leq \pi,$$

which is equivalent to

Theorem. *If* $H \in \mathcal{H}(Z(1))$ *and* $\sup(H) > \pi$,
then X_H *has a non-constant periodic orbit of period* 1.
(In particular, $H \notin \mathcal{H}_a(Z(1))$*).*

Beginning of proof: Assume that H vanishes near the origin. (If not, compose with appropriate symplectomorphism).

MAIN IDEA:

1. extend H to a function $\bar{H} \in C^\infty(\mathbb{R}^{2n}, \mathbb{R})$.
2. find the 1-periodic solutions of $\dot{z} = X_{\bar{H}}(z)$ as the critical points of the functional

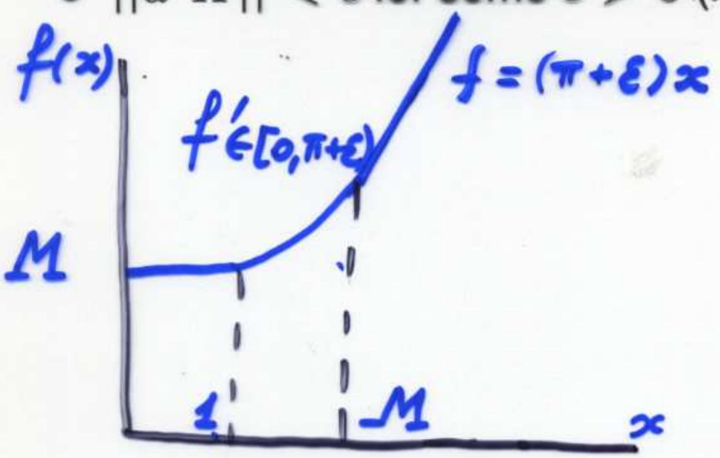
$$\Phi(z(t)) = \int_0^1 \left(\frac{1}{2} \langle -J\dot{z}, z \rangle - \bar{H}(z(t)) \right) dt$$

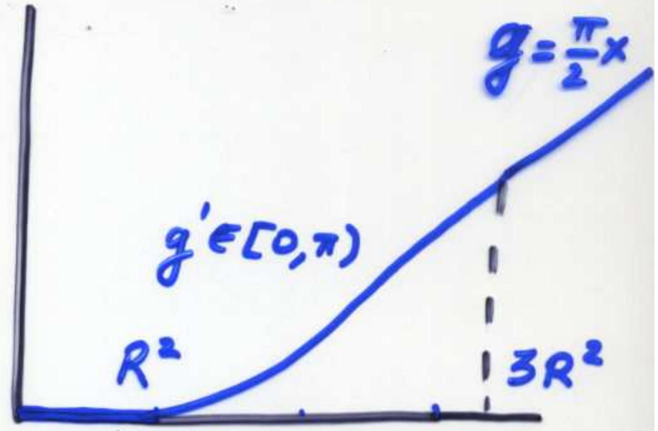
defined on the loop-space $C^\infty(S^1, \mathbb{R}^{2n})$.

Step 1: quadratic extension of $H \in \mathcal{H}(Z(1))$.

Extend $H \in \mathcal{H}(Z(1))$ to $\bar{H} \in C^\infty(\mathbb{R}^2)$, such that

- $\bar{H} = H$ on $Z(1) \cap B(R)$ for some R;
- $||d^2\bar{H}|| < c$ for some $c > 0$ (i.e., $\bar{H}$ has *quadratic growth*)

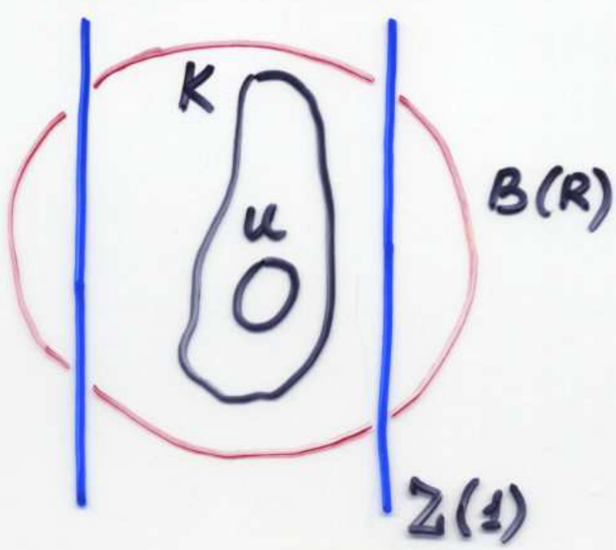




Let $z = (z_1, w)$, where $w = (z_2, \ldots, z_n)$.

$$\bar{H}(z) = \begin{cases} H(z), & z \in Z(1) \cap B(R), \\ f(|z_1|^2) + g(|w|^2), & z \notin Z(1) \cap B(R), \end{cases}$$

where $R > 1$ is such that $K \subset B(R)$.



Step 2: 1-periodic solutions with $\Phi(z(t)) > 0$

Lemma. *A 1-periodic solution $z(t)$ of $\dot{z} = X_{\bar{H}}(z)$ such that $\Phi(z(t)) > 0$ is non-constant and lies in K.*

Proof. If $z(t) = \text{const}$, then $\Phi(z(t)) < 0$ since $\bar{H} > 0$.

Claim: If $z(t_0) \subset \mathbb{R}^{2n} \setminus K$, then $\Phi(z(t)) \leq 0$.

Proof: Let $z(t)$ have $T = 1$, $\Phi(z) > 0$, $z(t_0) \in \mathbb{R}^{2n} \setminus K$. Then $\bar{H}(z(t_0)) > m(H)$, implying $\bar{H}(z(t)) > m(H)\ \forall t$. Hence, $z(t) \in \mathbb{R}^2 \setminus K$.
In this region, $\bar{H}(z) = f(|z_1|^2) + g(|w|^2)$, and

$$z_1(t) = e^{-2if'(|z_1|^2)} z_1(0), \quad w(t) = e^{-2ig'(|w|^2)} w(0).$$

Since $f' \in [0, \pi + \varepsilon)$, $g' \in [0, \pi)$, $\Phi(z) \leq -\varepsilon |z_1|^2 < 0$. $\square$

Thus, a 1-periodic solution of $\dot{z} = X_{\bar{H}}(z)$ with $\Phi(z(t)) > 0$ lies in K and is a solution of $\dot{z} = X_H(z)$.

Step 3: The analytical setting

Need to show that $\exists$ 1-periodic solution with $\Phi(z(t)) > 0$.

Extend $\Phi : C^\infty(S^1, \mathbb{R}^{2n}) \to \mathbb{R}$ to a Hilbert space.

Represent $z(t) \in C^\infty(S^1, \mathbb{R}^{2n})$ by $z(t) = \sum_{k\in\mathbb{Z}} e^{k\cdot 2\pi Jt} z_k$.

First term in $\Phi(z(t))$ is

$$a(z,z) = \int_0^1 \frac{1}{2}\langle -J\dot{z}, z\rangle\, dt = \pi \sum_{k\in\mathbb{Z}} k|z_k|^2,$$

can be extended to the Sobolev space $E = H^{1/2}(S^1)$

$$H^{1/2}(S^1) = \{z \in L^2(S^1) \;:\; \sum_{k\in\mathbb{Z}} |k|^{2\cdot\frac{1}{2}}|z_k|^2 < \infty\},$$

$$\langle z, z'\rangle = \langle z_0, z_0'\rangle + 2\pi \sum_{k\in\mathbb{Z}} |k| \cdot \langle z_k, z_k'\rangle.$$

$||\bar{H}(z)|| \leq M \cdot |z|^2$ implies that $b(z(t)) = \int_0^1 \bar{H}(z(t))dt$ extends to (a differentiable) functional on $L^2(S^1)$.

The extension $\Phi : E \to \mathbb{R}$ is given by $\Phi(z) = a(z) - b(z)$, with $\nabla\Phi(z) = z^+ - z^- - \nabla b(z)$.

Step 4: Regularity

Critical points of Φ are smooth periodic orbits:

Lemma. *A critical point $z(t) \in E$ of Φ lies in $C^\infty(S^1, \mathbb{R}^{2n})$, and is a 1-periodic solution of $\dot{z}(t) = X_H(z(t))$.*

Proof. Let z be a critical pt of Φ,

$$z = \sum_{k \in \mathbb{Z}} e^{k2\pi Jt} z_k, \quad \nabla H(z) = \sum_{k \in \mathbb{Z}} e^{k2\pi Jt} a_k.$$

Since $\nabla \Phi(z) = z^+ - z^- - \nabla b(z) = 0$, we have

$$\langle z^+ - z^-, v \rangle = \int_0^1 \langle \nabla H(z), v \rangle dt.$$

Choosing test functions $v(t) = e^{k2\pi Jt}$, we get

$$2\pi k z_k = a_k. \tag{1}$$

Thus,

$$\sum_{k \in \mathbb{Z}} |k|^2 |z_k|^2 \leq \sum |a_k|^2 < \infty,$$

$z \in H^1(S^1)$ implies $z \in C(S^1)$, which implies $\nabla H(z) \in C(S^1)$.

By (1), $z \in C^1(S^1)$ and solves Hamiltonian equation. Thus, $\nabla H(z) \in C^1(S^1)$.

Inductively, $z \in C^\infty(S^1)$. $\square$

Palais-Smale condition and Minimax

Critical pts of $\Phi : E \to \mathbb{R} \Leftrightarrow$ equilibria of $\dot{z} = -\nabla\Phi(z)$.

Definition. *Φ satisfies* PS-condition *if for $z_i \in E$ s.t.*

- $\nabla\Phi(z_i) \to 0$ *in E;*
- $|\Phi(z_i)| \leq c < \infty$ *for some $c \in \mathbb{R}$;*

has a convergent subsequence. Its limit is the critical pt.

Definition. *Let $\mathcal{F}$ be a family of sets $F \subset E$. The* minimax *of Φ on $\mathcal{F}$ is*

$$c(\Phi, \mathcal{F}) = \inf_{F\in\mathcal{F}} \sup_{x\in F} \Phi(x) \in \mathbb{R} \cup \{\pm\infty\}.$$

Lemma. *(Minimax Lemma) If*

1. *Φ satisfies PS-condition.*
2. *$\dot{z} = -\nabla\Phi(z)$ defines a global flow $\varphi^t(z)$.*
3. *If $F \in \mathcal{F}$, then $\varphi^t(F) \in \mathcal{F}\ \forall t \geq 0$ (positively invariant).*
4. *$c(\Phi, \mathcal{F})$ is finite,*

then $c(\Phi, \mathcal{F})$ is a critical value of Φ.

Step 5: Existence of a critical pt of $\Phi : E \to \mathbb{R}$.

1. Since Φ is smooth and $\nabla\Phi$ has linear growth, $\dot{z} = -\nabla\Phi(z)$ defines a unique global flow.
2. Φ satisfies the PS condition.
   (This follows from the asymptotic behavior of $\bar{H}$).

LINKING ARGUMENT (CONSTRUCTION OF $\mathcal{F}$):

Let $\Gamma_\alpha = \{z \in E^+ \mid ||z|| = \alpha\}$.

CLAIM. $\exists\alpha$: $\Phi_{|\Gamma_\alpha} \geq \beta > 0$.

Proof. Since $\bar{H}''(0) = 0$, it follows $\Phi''(0) = P^+ \oplus (-P^-)$. Since, $\Phi(0) = \Phi'(0) = 0$, it follows $\Phi_{|\Gamma_\alpha} \geq \beta > 0$. □

Let $\Sigma_\tau = \{z = z^- + z^0 + se \;:\; ||z^- + z^0|| \leq \tau,\; s \in [0, \tau]\}$, where $e = e(t) = (e^{2\pi i t}, 0, \ldots, 0) \in \mathbb{C}^n$.

CLAIM. $\exists\tau_0$ s.t. $\forall\tau \geq \tau_0$ $\Phi|_{\Sigma_\tau} \leq 0$.

Proof. Note that $\Phi_{|E^- \oplus E^0} \leq 0$. Since $\bar{H}(z) \leq (\pi + \varepsilon)|x_1|^2 + \frac{1}{2}\pi|w|^2 - C$, for $z \in \Sigma$

$$\Phi(z) \leq -\frac{1}{2}||z^-||^2 - \frac{1}{2}\pi|z^0|^2 - \varepsilon s^2 + C.$$

The claim follows for τ sufficiently large. □

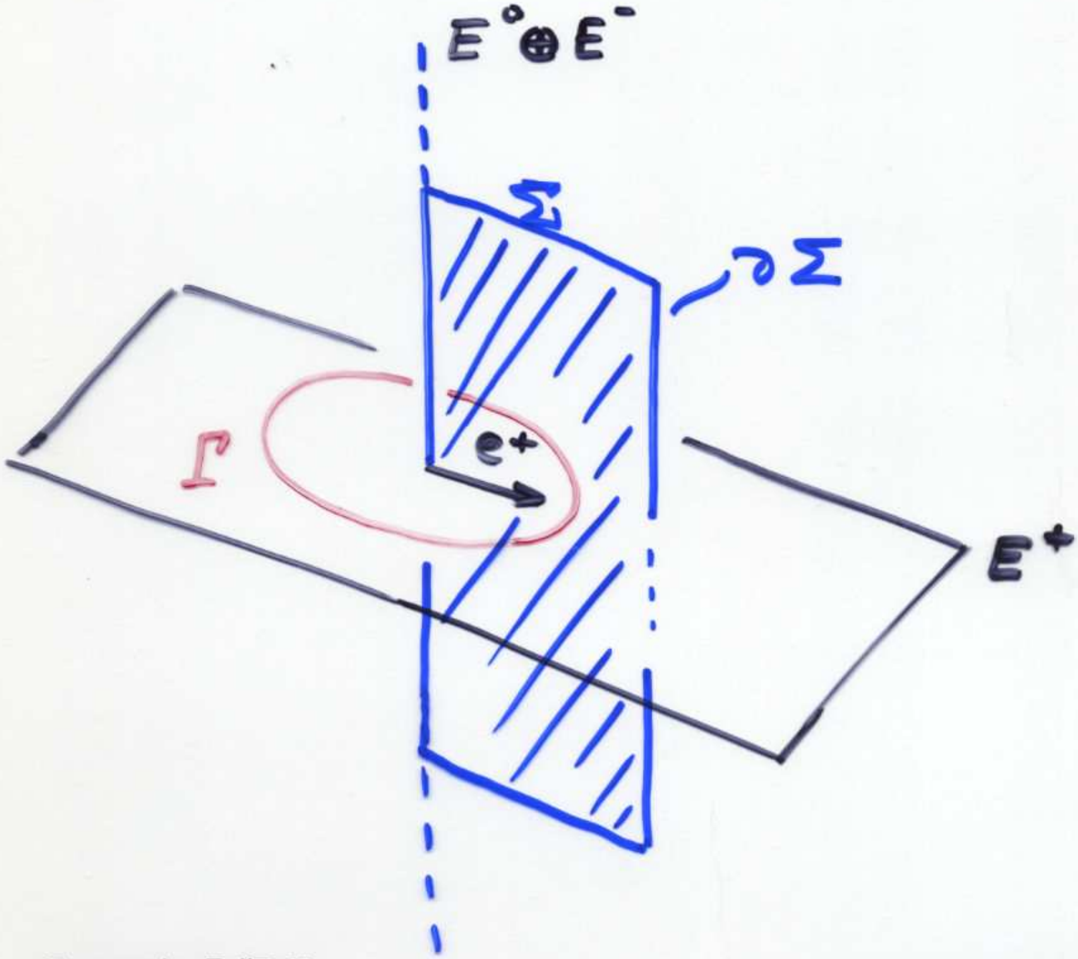
The Linking Argument

$\Phi(\varphi^t(z))$ decreases with $t \Rightarrow \Phi(\varphi^t(\partial\Sigma)) \leq 0 \; \forall \; t \geq 0$.

Thus, $\varphi^t(\partial\Sigma) \cap \Gamma = 0$.

Using compactness properties of the flow and the degree theory, this implies

$$\varphi^t(\Sigma) \cap \Gamma \neq 0, \quad \forall t \geq 0.$$



Applying the Minimax Lemma

Let $\mathcal{F} = \{\Phi = \varphi^t(\Sigma) \mid t \geq 0\}$.

Set $c(\Phi, \mathcal{F}) = \inf_{t\geq 0} \sup_{z\in\varphi^t(\Sigma)} \Phi(z)$.

Then

1. $\mathcal{F}$ is positive-invariant.
2. Since $\varphi^t(\Sigma) \cap \Gamma \neq 0$, $\inf_{z\in\Gamma} \Phi(z) \leq \sup_{z\in\varphi^t(\Sigma)} \Phi(z)$.
3. Since $\Phi_{|\Gamma} \geq \beta > 0$, it follows that $c(\Phi, \mathcal{F}) > -\infty$.
4. Since Φ maps bounded sets to bounded sets, $c(\Phi, \mathcal{F}) \leq \sup_{z\in\varphi^t(\Sigma)} \Phi(z) < \infty$.

Thus, by Minimax Lemma, $c(\Phi, \mathcal{F})$ is the critical value of Φ.

Let $z_0 \in E$ be such that $\Phi(z_0) = c(\Phi, \mathcal{F}) \geq \beta > 0$.

By regularity, $z_0 \in C^\infty(S^1, \mathbb{R}^{2n})$.

Thus, $c_0(Z(1)) < \pi$.

This finishes the proof that c_0 is a capacity function.

Capacity of convex domains

Recall that for an ellipsoid:

capacity= action of a special periodic orbit on the boundary.

Let C be compact convex region, $\partial C = M$.

Theorem. *(Rabinowitz, Weinstein). M has a periodic orbit.*

Hofer-Zehnder capacity=action of a special periodic orbit:

Theorem. *(Hofer-Zehnder)*

$c_0(C) = A(z_0(t)) = \inf\{|A(z(t))| : z(t) \in C^\infty(S^1, M)\}$,

where $A(z(t)) = \frac{1}{2}\int_0^T \langle -J\dot{z}, z\rangle dt$ is the reduced action.

Hypersurfaces which can't be made convex by a symplectic map

Proposition 1. *The Bordeaux bottle ∂M can not be mapped onto a convex hypersurface by a symplectomorphism.*

Proof. Let $B(r) \subset M \subset Z(r)$. Let C be a convex domain.
Then $c_0(M) = \pi r^2$.
Let $\varphi \in \mathrm{Symp}(\mathbb{R}^{2n})$ and $\varphi(\partial M) = \partial C$.

Invariance of capacity and action:

$$\begin{aligned} \pi r^2 &= c_0(M) = c_0(\varphi(M)) = c_0(C) \geq A(\gamma), \quad \forall \gamma \in \partial M, \\ \pi \rho^2 &= A(\gamma_0) = A(\varphi(\gamma_0)) < \pi r^2 \leq A(\varphi(\gamma_0)). \end{aligned}$$

Contradiction. □