SUMMER SCHOOL

Hamiltonian Mechanics and Integrable Systems

Lake Arrowhead

September 12 - 17, 2003

Compatible brackets in Hamiltonian mechanics

After H. P. McKean

Presented by Irina Nenciu - Caltech

Introduction and general results

Let d be fixed and let $C^\infty(\mathbb{R}^{2d})$ be a Lie algebra with the bracket

$$[H_1, H_2] = (\nabla H_1)^t J \nabla H_2 \tag{1}$$

where ∇ is the gradient and

$$J : \mathbb{R}^{2d} \to Gl(2d, \mathbb{R})$$

is smooth and skew. In particular, Jacobi's identity is satisfied:

$$[[H_1, H_2], H_3] + [[H_2, H_3], H_1] + [[H_3, H_1], H_2] = 0 \tag{2}$$

for any $H_1, H_2, H_3 \in C^\infty(\mathbb{R}^{2d})$.

Note that Jacobi's identity (2) is equivalent to any of the following conditions on J:

- For all triples (i, j, k)

$$(J\nabla)_i J_{jk} + (J\nabla)_j J_{ki} + (J\nabla)_k J_{ij} = 0; \quad (3)$$

- For all triples (i, j, k)

$$\frac{\partial J^{-1}_{ij}}{\partial x_k} + \frac{\partial J^{-1}_{jk}}{\partial x_i} + \frac{\partial J^{-1}_{ki}}{\partial x_j} = 0; \quad (4)$$

- $\sum_{i<j} J^{-1}_{ij} dx_i \wedge dx_j$ is closed.

Assume that there exist two such brackets

$$[H_1, H_2]_J = (\nabla H_1)^t J \nabla H_2$$
$$[H_1, H_2]_K = (\nabla H_1)^t K \nabla H_2$$

with J and K satisfying the conditions on the previous slides.

Definition 1 *We say that a Hamiltonian H_0 can be raised if there exists another Hamiltonian H_1 so that*

$$[f, H_0]_K = [f, H_1]_J \tag{5}$$

for all $f \in C^\infty(\mathbb{R}^{2d})$. In this case we say that H_0 raises to H_1 or that H_1 lowers to H_0, and denote these relations by $H_0 \uparrow H_1$ and $H_1 \downarrow H_0$, respectively.

We immediately obtain the following

Lemma 2 *Consider* J *and* K *as above and assume that there exists a sequence of Hamiltonians so that* $H_0 \uparrow H_1 \uparrow H_2 \uparrow \ldots$. *Then*

$$[H_i, H_j]_J = [H_i, H_j]_K = 0 \tag{6}$$

for all $i, j \geq 0$.

Notice that, for $i < j$,

$$\begin{aligned} [H_i, H_j]_J = [H_i, H_{j-1}]_K = -[H_{j-1}, H_i]_K \\ = -[H_{j-1}, H_{i+1}]_J = [H_{i+1}, H_{j-1}]_J. \end{aligned}$$

We have

$$[H_i, H_j]_J = [H_{i+1}, H_{j-1}]_J.$$

Then

- If i and j have the same parity, then after a finite number of steps of the above computation we get

  $$[H_i, H_j]_J = [H_k, H_k]_J = 0$$

  where $k = \frac{i+j}{2}$.

- If i and j have different parities, then

  $$[H_i, H_j]_J = [H_j, H_i]_J = 0.$$

So $[H_i, H_j]_J = 0$ for all i, j. This immediately implies that $[H_i, H_j]_K = 0$ for all i, j.

Question: When is it possible to have such infinite sequences of raising Hamiltonians $H_0 \uparrow H_1 \uparrow H_2 \uparrow \ldots$?

Want a condition on J and K so that

$$H_0 \uparrow H_1 \quad \Rightarrow \quad H_1 \uparrow ?$$

for all H_0.

This in turn is implied by the condition that $\uparrow = \downarrow$. So the next Lemma gives the natural condition to answer our question.

Lemma 3 *If J, K and $J + K$ all produce honest brackets (i.e. the brackets they define obey the Jacobi identity), then the class of Hamiltonians that can be raised coincides with the class of those that can be lowered.*

In this case we say that J and K are compatible.

Idea of the proof. The condition $\uparrow=\downarrow$ is equivalent to

$$J^{-1}K\nabla H \text{ is a gradient}$$

$$\Updownarrow$$

$$K^{-1}J\nabla H \text{ is a gradient}$$

which is further equivalent to the fact that both or neither of the matrices

$$A_{+}(H) = \left[\frac{\partial}{\partial x_i}(J^{-1}K\nabla H)_j\right]_{1\le i,j\le 2d}$$

and

$$A_{-}(H) = \left[\frac{\partial}{\partial x_i}(K^{-1}J\nabla H)_j\right]_{1\le i,j\le 2d}$$

are symmetric.

This last condition is implied by the relation

$$K(A_-(H) - A_-(H)^t)K + J(A_+(H) - A_+(H)^t)J = 0. \tag{7}$$

Using the fact that J and K satisfy the Jacobi identity, one gets that equation (7) is equivalent to the fact that $J + K$ also satisfies the Jacobi identity.

Remark. Let $L(c) = J - cK$. Then notice that

$$\begin{aligned}(2)_{L(c)} = (2)_J + c^2 \cdot (2)_{\mathsf{K}} \\ - c[(2)_{J+K} - (2)_J - (2)_K]\end{aligned}$$

is a quadratic polynomial in c.

So J and K are compatible iff $L(c)$ satisfies the Jacobi identity for any c iff $L(c)$ satisfies the Jacobi identity for 3 values of c.

Recall that, if $L(c)$ is invertible, then it obeys the Jacobi identity iff $L(c)^{-1}$ defines a closed 2-form. For c small enough

$$L(c)^{-1} = \sum_{n \geq 0} c^n (J^{-1}K)^n J^{-1}.$$

Hence if J and K are compatible, then $(J^{-1}K)^n J^{-1}$ define closed 2-forms for all $n \geq 0$. Conversely, it turns out that if $(J^{-1}K)^n J^{-1}$ define closed 2-forms for $n = 0, 1, 2$, then J and K are compatible.

Magri's result

Want to find a second, compatible, bracket for a given completely integrable Hamiltonian system.

Assume first that such a bracket exists and find a necessary form for it.

Hypothesis 1. J and K give two compatible brackets.

Let $\pi_k = \mathrm{Tr}(J^{-1}K)^k$ for all $k \geq 1$. Then one can prove that

$$\pi_1 \uparrow \frac{\pi_2}{2} \uparrow \frac{\pi_3}{3} \uparrow \ldots \qquad (8)$$

and, consequently, the π_k's commute.

To prove this, take λ large enough so that $L = \lambda J - K$ is invertible. Then

$$
\begin{aligned}
(L\nabla \log\det L)_i &= \sum_j L_{ij} \mathrm{Tr}\Big(L^{-1}\frac{\partial L}{\partial x_j}\Big) \\
&= \sum_{j,a,b} L_{ij} L^{-1}_{ab} \frac{\partial L_{ba}}{\partial x_j} \\
&= -\sum_{a,b,j} L^{-1}_{ab}\Big(L_{bj}\frac{\partial L_{ai}}{\partial x_j} + L_{aj}\frac{\partial L_{ib}}{\partial x_j}\Big) \\
&= -\sum_{a,j} \delta_{aj}\frac{\partial L_{ai}}{\partial x_j} + \sum_{b,j}\delta_{jb}\frac{\partial L_{ib}}{\partial x_j} \\
&= 2\sum_j \frac{\partial L_{ij}}{\partial x_j}
\end{aligned}
$$

for all i.

Remark. The third identity uses the fact that L satisfies the Jacobi identity for any value of λ.

Now sum over i and expand in λ. The right-hand side is linear in λ. The left hand side can be expanded for large enough λ as follows:

$$L\nabla\Big(2d\log\lambda + \log\det J + \log\det(I - \lambda^{-1}J^{-1}K)\Big)$$

$$= L\nabla \log\det J - L\nabla \mathrm{Tr}\log(I - \lambda^{-1}J^{-1}K)$$

$$= L\nabla \log\det J - L\nabla \mathrm{Tr}\Big(\sum_{n\geq 1}\lambda^{-n}\frac{(J^{-1}K)^n}{n}\Big)$$

$$= L\nabla \log\det J - \sum_{n\geq 0}\lambda^{-n} J\nabla \frac{\mathrm{Tr}(J^{-1}K)^{n+1}}{n+1}$$

$$+ \sum_{n\geq 1}\lambda^{-n} K\nabla \frac{\mathrm{Tr}(J^{-1}K)^{n}}{n}$$

So all the coefficients of λ^{-n} for $n \geq 1$ must be zero:

$$J\nabla\left(\frac{\pi_{n+1}}{n+1}\right) = K\nabla\left(\frac{\pi_n}{n}\right)$$

for all $n \geq 1$. This is exactly the definition of

$$\frac{\pi_n}{n} \uparrow \frac{\pi_{n+1}}{n+1}.$$

In particular we can conclude from Lemma 2 that

$$[\pi_n, \pi_m]_J = [\pi_n, \pi_m]_K = 0$$

for all $n, m \geq 1$.

Hypothesis 2. The spectrum of $J^{-1}K$ is *ample*; that is the sense that the gradients $\nabla\pi_1, \cdots, \nabla\pi_d$ are independent.

In this case, fixing the values of the π_k's defines a smooth d-dimensional manifold $M \subset \mathbb{R}^{2d}$ on which the $\nabla\pi_k$'s are normal and the $J\nabla\pi_k$'s are tangent for all $1 \leq k \leq d$:

$$\langle \nabla\pi_k, J\nabla\pi_l \rangle = (\nabla\pi_k)^t J\nabla\pi_l = [\pi_k, \pi_l]_J = 0.$$

- If $H \in \uparrow = \downarrow$, then $[\pi_k, H]_J = [\pi_k, H]_K = 0$.

- If the spectrum of $J^{-1}K$ is ample, then it is double. Let $\lambda_1, \ldots, \lambda_d$ denote the eigenvalues of $J^{-1}K$.

- $H \in \uparrow = \downarrow$ if and only if $H = \sum_j h_j(\lambda_j)$.

Under

Hypothesis 1. J and K give two compatible brackets

and

Hypothesis 2. The gradients $\nabla\pi_1, \cdots, \nabla\pi_d$ are independent

we have that the functions $H_k = \frac{\pi_k}{k}$ $(1 \leq k \leq d)$ commute and can be completed by functions G_k $(1 \leq k \leq d)$, canonically conjugate relative to J.

Now we can compute the K-bracket of the H's and G's.

$$[H_i, H_j]_K = [H_i, H_{j+1}]_J = 0.$$

$$\begin{aligned} [H_i, G_j]_K &= [H_{i+1}, G_j]_J \\ &= \begin{cases} 1, & \text{if } j = i+1 \leq d; \\ 0, & \text{if } j \neq i+1 \leq d; \\ \frac{\partial H_{d+1}}{\partial H_j}, & \text{if } i = d. \end{cases} \end{aligned}$$

Finally, using the Jacobi identity for $[\cdot,\cdot]_K$ and the fact that $[H_i, G_j]_J = \delta_{ij}$ is a constant, we get

$$\begin{aligned} [[G_i, G_j]_K, H_k]_J &= [[G_i, G_j]_K, H_k^-]_K \\ &= -[[G_j, H_k^-]_K, G_i]_K \\ &\quad - [[H_k^-, G_i]_K, G_j]_K \\ &= -[[G_j, H_k]_J, G_i]_K \\ &\quad - [[H_k, G_i]_J, G_j]_K \\ &= 0 \end{aligned}$$

and hence

$$[G_i, G_j]_K = c_{ij}(H_1, \ldots, H_d).$$

Since, by Hypothesis 2, ∇H_k and ∇G_k ($1 \leq k \leq d$) span $\mathbb{R}^{2d}$, we find that K is uniquely determined by Magri's rule and it equals

$$\sum_{i<j} c_{i,j}(J\nabla H_i \otimes J\nabla H_j - J\nabla H_j \otimes J\nabla H_i) + \sum_{j=1}^{d} (J\nabla H_j^+ \otimes J\nabla G_j - J\nabla G_j \otimes J\nabla H_j^+)$$

Let us remember that, if $v, w \in \mathbb{R}^{2d}$, then

$$v \otimes w : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$$

by

$$(v \otimes w)u = v\langle w, u\rangle.$$

So Magri's rule is just telling us that

$$K\nabla H_j = J\nabla H_{j+1} \quad \text{for all}\, j \geq 1 \tag{9}$$

and

$$K\nabla G_j = \begin{cases} J\nabla G_{j-1} + \sum_{i\neq j} J\nabla H_i, & \text{if}\, j \geq 2; \\ \sum_{i\neq j} J\nabla H_i, & \text{if}\, j = 1. \end{cases} \tag{10}$$

Yet another way of expressing this is saying that the matrix $J^{-1}KJ^{-1}$ produces the 2-form

$$\sum_{j=1}^{d} dH_{j+1} \wedge dG_j + \sum_{i<j} c_{ij} dH_i \wedge dH_j.$$

So if we define

$$c_- = \sum_{i<j} c_{ij}\, dH_i \wedge dH_j$$

we get

Equivalence 1.1 $J^{-1}KJ^{-1}$ defines a closed 2-form if and only if c_- is closed.

Similarly, the form defined by $(J^{-1}K)^2J^{-1}$ is

$$\sum_{j=1}^{d} dH_{j+2} \wedge dG_j + c_+$$

where

$$c_+ = \sum_{i<j} (dH_i^+ \wedge dH_j + dH_i \wedge dH_j^+).$$

Equivalence 1.2 $(J^{-1}K)^2J^{-1}$ defines a closed 2-form if and only if c_+ is closed.

Remark. To prove the formulae for $J^{-1}KJ^{-1}$ and $(J^{-1}K)^2J^{-1}$ we also use the fact that $H_j = p_j(H_1, \dots, H_d)$ for all $j \geq d+1$.

So when can one construct a second bracket for a completely integrable system?

Hypothesis. Let H_i, $i \geq 1$ be any family of commuting functions with respect to a bracket J. Assume that:

- G_j, $1 \leq j \leq d$ is a family of functions canonically paired to the H_i, $1 \leq i \leq d$;
- $H_j = p_j(H_1, \ldots, H_d)$ for all $j \geq d+1$;
- $[c_{ij}(H_1, \ldots, H_d)]_{1 \leq i,j \leq d}$ is such that both forms c_- and c_+ defined above are closed.

Conclusion. The skew form K produced by Magri's rule is compatible with J. Moreover, $H_1 \uparrow H_2 \uparrow \ldots$.

Further remarks.

- Essentially, $H_i = \frac{\pi_i}{i}$;

- The gauge c_{ij} can be removed by an appropriate choice of the G_j's.

Big Conclusion. If $\dot{x} = J\nabla H(x)$ is an integrable flow on $\mathbb{R}^{2d}$ with d independent commuting integrals of motion $H = H_1, \dots, H_d$, then Magri's formula provides a compatible K so that

$$H_1 = \pi_1 \uparrow H_2 = \frac{\pi_2}{2} \uparrow \dots .$$

Moreover, by wisely choosing the H's and the G's, one remove the c_{ij} and make

$$J^{-1}K\nabla\lambda_i = \lambda_i\nabla\lambda_i, \quad J^{-1}K\nabla\mu_i = \lambda_i\nabla\mu_i.$$