SUMMER SCHOOL

Hamiltonian Mechanics and Integrable Systems

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Instability of dynamical systems with several degrees of freedom

After V. I. Arnol'd

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Recall that a Hamiltonian system is a system of $2d$ equations

$$\dot{p}_j = -\frac{\partial H}{\partial q_j} \quad \text{and} \quad \dot{q}_j = \frac{\partial H}{\partial p_j}$$

for $1 \leq j \leq d$.

If the system is completely integrable, then there exists $(I, \phi) \in \mathbb{R}^d \times \mathbb{T}^d$, called action-angle variables, in which $H(I, \phi) = H_0(I)$.

Therefore the equations of motion reduce to

$$\dot{I}_j = 0 \quad \Rightarrow \quad I_j = \text{const.}$$

and

$$\dot{\phi}_j = \frac{\partial H}{\partial I_j} \equiv \omega_j(I).$$

and the motion of the system is quite simple:

- $I_j(t) = I_j(0)$ are constants of motion;
- $\phi_j(t) = t\omega_j + \phi_j(0)$ describes the evolution of the trajectory along a d-dimensional torus.

We are interested in the case where a perturbation is added to H_0:

$$H(I,\phi) = H_0(I) + \epsilon H_1(I,\phi). \qquad (1)$$

We follow Lochak [L] to present the 3 types of problems related to the behavior of systems as in (1).

- *Classical perturbation theory*: stability of the action variables over long intervals of time.

  Given initial conditions $(p(0), q(0))$, one seeks a bound on the drift $\|I(t) - I(0)\|$ for $|t| \le t(\epsilon)$ large w.r.t. $1/\epsilon$.

  The most important such estimate is the Nekhoroshev estimate. In particular $t(\epsilon)$ is exponential in $1/\epsilon$.

- *Geometric perturbation theory*: the search for geometric objects invariant under the flow of (1).

  KAM theory proves the existence of 'many' invariant tori of dimension $\leq d$; other objects, like invariant stable and unstable manifolds associated to lower dimensional KAM tori have also been discovered.

- *Arnol'd diffusion*: everything else!

  More precisely, what happens to trajectories living outside the invariant geometric objects?

  Basic question: find trajectories such that for some time t the drift $\|I(t) - I(0)\|$ is of order 1.

Arnol'd's example.

$$\Omega = \mathbb{R}^2 \times \mathbb{T}^3,$$

with $(I_1, I_2) \in \mathbb{R}^2$ and $(\phi_1, \phi_2, t) \in \mathbb{T}^3$.

The Hamiltonian is given by

$$H = \frac{1}{2}(I_1^2 + I_2^2) + \epsilon(\cos\phi_1 - 1)(1 + \mu B),$$

where

$$B = \sin\phi_2 + \cos t.$$

In other words, we consider the system of differential equations

$$\dot{\phi}_j = I_j, \tag{2}$$

$$\dot{I}_1 = \epsilon \sin\phi_1(1 + \mu B),$$

$$\dot{I}_2 = \epsilon(1 - \cos\phi_1)\mu\cos\phi_2$$

with $j = 1, 2$ and $0 < \mu \ll \epsilon \ll 1$.

Theorem 1 *Assume* $0 < A < B$. *For every* $\epsilon > 0$ *there exists a* $\mu_0 = \mu_0(A, B, \epsilon) > 0$ *such that for* $0 < \mu < \mu_0$ *the system* (2) *has a solution satisfying* $I_2(0) < A$, $I_2(t) > B$ *for a certain* t.

The Arnol'd mechanism. The mechanism proposed by Arnol'd involves 3 general steps.

Step 1. Existence of invariant "whiskered" tori.

Step 2. Transverse intersections.

Step 3. Transition chains.

Definitions and Step 3.

Assume that in the phase space of a dynamical system there is an invariant torus T and on it a quasi-periodic motion with dense orbits.

We shall call T a **whiskered torus** if

- T is a connected component of the intersection of two invariant, open manifolds Y^- and Y^+;

- All the trajectories of the **arriving whisker** Y^- approach T as $t \to \infty$

$$\lim_{t\to\infty} |\zeta(t) - T| = 0 \quad \text{for} \quad \zeta(0) \in Y^-;$$

- On the **departing whisker** Y^+ all the trajectories approach T as $t \to -\infty$

$$\lim_{t\to-\infty} |\zeta(t) - T| = 0 \quad \text{for} \quad \zeta(0) \in Y^+.$$

Example: the standard whiskered torus.

We consider the system

$$\dot{x} = \lambda x, \quad \dot{y} = -\mu y, \quad \dot{z} = 0, \quad \dot{\phi} = \omega$$

in $\mathbb{R}^{l_+} \times \mathbb{R}^{l_-} \times \mathbb{R}^{l_0} \times \mathbb{T}^k$, with $\lambda, \mu > 0$ and ω nonresonant.

Then

$$T = \{x = y = z = 0\}$$

is a k-dimensional invariant torus and

- $Y^+ = \{y = z = 0\}$ is the $(l_+ + k)$-dimensional departing whisker (or 'unstable manifold' M^u).

- $Y^- = \{x = z = 0\}$ is the $(l_- + k)$-dimensional arriving whisker (or 'stable manifold' M^s).

Consider M and Ω submanifolds of a space X. We say that Ω **obstructs** M at $x \in M$ if

$$\Omega \cap N \neq \emptyset$$

for all submanifolds N that intersect M transversely at x.

Example.

Let T be a whiskered torus. We call T a **transition torus** if

for any $\xi \in Y^-$ and any neighborhood U of ξ

we have that

$\Omega = \cup_{t>0} U(t)$ obstructs Y^+ at an arbitrary point η.

Example. The standard whiskered torus is a transition torus.

Let $\xi = (0, y_0, 0, \phi_0)$ and $\eta = (x_1, 0, 0, \phi_1)$. As ω is nonresonant, there exists a sequence $t_i \to \infty$ such that

$$|(\phi_0 + t_i\omega) - \phi_1| \to 0.$$

As $\lambda > 0$ and $\mu > 0$, we have that

$$x^{(j)}(t) = e^{\lambda t} x^{(j)}(0) \to x^{(j)}(0) \cdot \infty$$

and

$$y^{(j)}(t) = e^{-\mu t} y^{(j)}(0) \to 0$$

as $t \to \infty$.

Assume that a dynamical system has transition tori $T_1, \dots, T_s$. These tori form a **transition chain** if Y_i^+ is transverse to Y_{i+1}^- at some point of their intersection for all $i = 1, \dots, s-1$.

Lemma 2 *Let $T_1, \dots, T_s$ be a transition chain. Then an arbitrary neighborhood U of an arbitrary point $\xi \in Y_1^-$ is connected with an orbit $\zeta(t)$ to an arbitrary neighborhood V of an arbitrary point $\eta \in Y_s^+$.*

Proof. Let $\Omega = \bigcup_{t>0} U(t)$. Since T_1 is a transition torus, Ω obstructs Y_1^+ at

$$\xi_1 \in Y_1^+ \cap Y_2^-;$$

therefore there exists

$$\xi_1' \in \Omega \cap Y_2^-.$$

Let U_1 be a neighborhood of ξ_1' contained in Ω. Then $\Omega_1 = \bigcup_{t>0} U_1(t) \subset \Omega$ and we repeat.

Thus we conclude that Ω obstructs Y_s^+ at η.

Step 1. Invariant tori.

The unperturbed case: $\epsilon = 0$.

In this case

$$H = \frac{1}{2}(I_1^2 + I_2^2)$$

and the system is completely integrable.

Every torus

$$I_1 = \omega_1 \quad , \quad I_2 = \omega_2$$

is invariant and on it three-frequency motion takes place:

$$\dot{\phi}_1 = \omega_1, \quad \dot{\phi}_2 = \omega_2, \quad \dot{t} = 1$$

with $n_1\omega_1 + n_2\omega_2 + n_0 \neq 0$ for all $\overrightarrow{n} \in \mathbb{Z}^3 \backslash 0$.

Case $\epsilon \neq 0$ and $\mu = 0$.

In this case we can separate variables

$$H = H^{(1)} + H^{(2)}$$

with

$$H^{(1)} = \frac{1}{2} I_1^2 + \epsilon(\cos \phi_1 - 1), \quad H^{(2)} = \frac{1}{2} I_2^2.$$

Then the torus defined by

$$T_\omega = \{I_1 = \phi_1 = I_2 - \omega = 0\}$$

is a 2-dimensional whiskered torus of the system (2). The whiskers are given by

$$I_1 = \pm 2\sqrt{\epsilon} \sin \frac{\phi_1}{2}, \quad I_2 = \omega.$$

Case $\epsilon \neq 0$ and $\mu \neq 0$.

Notice that the tori

$$T_\omega = \{I_1 = \phi_1 = I_2 - \omega = 0\}$$

remain invariant!

For $\mu = 0$, the 2 whiskers form just one manifold. When μ becomes non-zero, this manifold splits into two whiskers. The fact that they survive can be proved using the theory of contractive maps, but it is just a reflection of the hyperbolic character of the invariant tori.

Step 2. Transverse intersections of the whiskered tori.

Lemma 3 *Assume* $\tilde{A} < \omega < \tilde{B}$*. Then there exists a constant* $\kappa = \kappa(\tilde{A}, \tilde{B}, \epsilon, \mu) > 0$ *so that for all* $|\omega' - \omega| < \kappa$ *we have that* $Y_{\omega}^{+} \cap Y_{\omega'}^{-} \neq \emptyset$*.*

So we can choose a transition chain of tori

$$T_{\omega_1}, \ldots, T_{\omega_s}$$

with

$$I_2(T_{\omega_1}) = \omega_1 < A \text{ and } I_2(T_{\omega_s}) = \omega_s > B.$$

Then there exists a trajectory $\zeta(t)$ that passes arbitrarily close to both T_{ω_1} and T_{ω_s}; on this trajectory, I_2 evolves from being less than A to being greater than B in finite time.

Proof of the Lemma.

The equation of the non-perturbed whiskers is

$$H^{(1)} = 0, \quad H^{(2)} = \frac{\omega^2}{2}.$$

Let $\alpha > 0$. For $|\phi_1| < 2\pi - \alpha$ and $|\phi_1 - 2\pi| < 2\pi - \alpha$, respectively, the eqns. of $Y_\omega^\pm$ can be written

$$\begin{aligned} H^{(1)} &= \Delta_1^\pm(\phi_1; \phi_2, t; \omega), \\ H^{(2)} &= \frac{\omega^2}{2} + \Delta_2^\pm(\phi_1; \phi_2, t; \omega) \end{aligned}$$

with $\Delta_{1,2}^\pm = O(\mu)$.

We want to find a point in the intersection $Y_{\omega}^{+} \cap Y_{\omega'}^{-}$ with $\phi_1 = \pi$. This reduces to solving the system

$$\Delta_1^{+}(\pi; \phi_2, t; \omega) = \Delta_1^{-}(\pi; \phi_2, t; \omega')$$

$$\frac{\omega^2}{2} + \Delta_2^{+}(\pi; \phi_2, t; \omega) = \frac{\omega'^2}{2} + \Delta_2^{-}(\pi; \phi_2, t; \omega').$$

But it is sufficient to solve this system to first order in μ. Set

$$\Delta_{1,2}^{\pm} = \mu\delta_{1,2}^{\pm} + O(\mu^2).$$

We want

$$\delta_1^{+} - \delta_1^{-} = 0, \quad \mu(\delta_2^{+} - \delta_2^{-}) = \frac{1}{2}(\omega^2 - \omega'^2).$$

The issue is resolved as one can explicitly compute

$$\delta_1 = \delta_1^+ - \delta_1^- = 2\pi \sinh^{-1}\left(\frac{\pi}{2\sqrt{\epsilon}}\right) \sin t$$

$$\delta_2 = \delta_2^+ - \delta_2^- = 2\pi\omega^2 \sinh^{-1}\left(\frac{\omega\pi}{2\sqrt{\epsilon}}\right) \cos\phi_2.$$

For $t = 0$ one can solve the approximate system for

$$|\omega^2 - \omega'^2| < 4\pi\mu\omega^2 \sinh^{-1}\left(\frac{\omega\pi}{2\sqrt{\epsilon}}\right) \approx \mu e^{-1/\sqrt{\epsilon}}.$$

Bibliography.

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