Conformal and Quasiconformal Maps

Summer School

September 15.-20. 2000

The purpose of the Summer School is twofold: on the one hand participants will learn about the mathematical subject, which is conformal and quasiconformal maps. On the other hand, the School is addressed to young mathematicians in the process of becoming independent researchers, so the program is designed to give a maximum of insight into what mathematical research is like nowadays. Thus we will read a number of recent papers/preprints, and some written with the participation of young researchers.

This also means that we will not devellop every detail of the theory, and we will have to take some of the results and implications quoted in the papers as granted without full explanation.

We give a brief outline of the topics, the numbers are according to the list of topics for the summer school.

A) For us the most important example of a conformal map is the Riemann map, i.e. a holomorphic and bijective map, from some open domain in the plane to the unit disc. More precisely, there is a three parameter family of such maps, but for this outline we will not worry about this. Thurston suggested a discrete algorithm to approximate this Riemann mapping by taking hexagonal circle mappings of the domain and map it to combinatorially equivalent cirle packing of the disc. Topics 1) and 2) will discuss the convergence of this discrete algorithm to the actual Riemann map.

B) A quasiconformal map is a differentiable map from a planar domain to a planar domain, such that the Jacobian has bounded excentricity (quotient of larger to smaller eigenvalue). For conformal maps the excentricity is constant one, so quasiconformal maps are generalizations of conformal maps. The theory of quasiconformal maps has (historically a celebrated insight) links to the theory of dynamical systems, and we will focus on that aspect of quasiconformal maps. In 3), quasiconformal maps are used to prove a theorem in dynamical systems (non-existence of wandering domains for rational iteration). Vice versa, in 4) an important regularity result is proven for quasiconformal maps which uses some insights from the theory of dynamical systems. The original paper in 4) used some ingredient about holomorphic motion, which is very interesting for its own sake and we will be discuss it in 5)

C)The Löwner equation describes a continuous deformation from the identity on the disc to the Riemann map from the unit disc to a domain (from now on the inverse of the map discussed above). It was invented to study the Bieberbach conjecture, now de Branges' theorem, which gives very precise bounds on the power series coefficients of the Riemann map. We will read about Bieberbach's conjecture in 6).

D) The Löwner equition has as input a certain "driving force" which describes the continuous deformation of the Riemann map. The deformation of the Riemann map is particularly visualizable if the family of domains that the Riemann mappings are onto are slit discs where the slit is growing from the boundary as the Lowner equation is evolving. Topic 7) describes regularity properties of the mapping from the driving force function (Hölder continuous with exponent 1/2, which is close to Brownian motion) to the slit function. If the driving force is Brownian motion, the Löwner equation is called a stochastic Löwner equation. Then the slit (which sometimes is not really a slit) describes a family of very important random processes with conformal invariance. The basics of stochastic Löwner equations are discussed in topics 8) and 9). There is a one parameter family of such stochastic Löwner equations, and the qualitative behaviour of the slits has several phase transitions as the parameter changes. Here and in topic 10 we will use a few basic concepts of stochastic integration.

E) Finally we discuss some random processes that appear in the context of the stochastic Löwner equation. Topic 10) discusses the loop erased random walk, one of the important conformally invariant random process that can be described by the stochastic Löwner equation. Topic 11) discusses percolation on the triangular grid and shows its conformal invariance. (This has some of the spirit of the circle packings discussed above in that the combinatorics of a triangular grid has a very rigid structure in the plane.)