Spectral theory of 1D Schrödinger operators

Summer School

September 10.-15. 2000

The purpose of the Summer School is twofold: on the one hand participants will learn about the mathematical subject, which is spectral theory of Schrödinger operators in one dimension. On the other hand, the School is addressed to young mathematicians in the process of becoming independent researchers, so the program is designed to give a maximum of insight into what mathematical research is like nowadays. Thus we will read a number of recent papers/preprints, none more than 3 years old, and some written with the participation of very young researchers.

This also means that we will not devellop every detail of the theory, and we will have to take some of the results and implications quoted in the papers as granted without full explanation.

The one dimensional Schrödinger operator is given by

$$Af(x) = -f''(x) + V(x)f(x)$$

Spectral theory of this operator is largely about eigenvalues and eigenfunctions of this operator. The selected papers circle around three different aspects of this theory:

1) Christ/Kiselev prove that if V is in $L^p(\mathbf{R})$ with $1 \leq p < 2$, then the Schrödinger operator A has two linear independent *bounded* eigenfunctions for almost every positive eigenvalue λ. This implies that the absolute continuous spectrum of A is essentially supported by $[0, \infty)$. Deift/Killip prove this corollary by different means for even more general V, e.g. $V \in L^2(\mathbf{R})$.

2) Bourgain/Goldstein and Jitomirskaya consider a discrete variant of the Schrödinger operator:

$$Bf(n) = f(n+1) + f(n-1) + V(n)f(n)$$

Jitomirskaya considers a potential of the form $V(n) = \lambda \cos(\omega n + \theta)$ with irrational ω (quasi periodic). More precisely she shows that if $\lambda > 2$ and ω is Diophantine, then for almost all θ the operator B has pure point spectrum and exponentially decaying eigenfunctions (Anderson localization). Bourgain/Goldstein show a similar result by different methods: they can handle arbitrary trigonometric polynomials rather than cos, but have less specific control over ω.

3) The remaining papers (Laptev, Weidl, Hundertmark, Lieb, Thomas, Benguria, Loss) focus on Lieb-Thierring inequalities and best constans therein. If $E_1, E_2, \ldots$ are the negative eigenvalues of A, then a Lieb Thirring inequality is one of the form

$$\sum_i |E_i|^\gamma \leq C \int |V_-(x)|^p \, dx$$

for some γ, p, and C. Here V_- is the negative part of V, i.e., $V_- = \min(V, 0)$. Such inequalities play a role in physics when counting bounded states of a system. The papers we read discuss best constants C (possibly infinite) for given γ and p.