Some open problems in mathematics

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These are some of my favorite open problems in mathematics.

1 Tri-linear Hilbert transform

Let α be an irrational number. For compactly supported smooth functions f_1, f_2, f_3, f_4 on $\mathbf{R}$ define

$$\Lambda(f_1, f_2, f_3, f_4) = \int_{\mathbf{R}} p.v. \int_{\mathbf{R}} f_1(x-t) f_2(x+t) f_3(x-\alpha t) \frac{dt}{t} f_4(x) dx$$

Here the principal value is defined as

$$p.v. \int_{\mathbf{R}} \dots \frac{dt}{t} = \lim_{\epsilon \to 0} \int_{\mathbf{R} \setminus [-\epsilon, \epsilon]} \dots \frac{dt}{t}$$

Prove or disprove (here $\|f\|_4$ denotes the L^4 norm)

Conjecture 1 *There is a constant C independent of $f_1, \dots, f_4$ such that*

$$|\Lambda(f_1, f_2, f_3, f_4)| \leq C \|f_1\|_4 \|f_2\|_4 \|f_3\|_4 \|f_4\|_4$$

To trace some background information start with [2].

2 Non-linear Carleson theorem

Let V be a function in $L^2(\mathbf{R})$. Then for $k \geq 0$, by elementary methods, the ordinary differential equation

$$f'' + Vf = -kf$$

has a two dimensional space of classical solutions (with absolutely continuous derivatives) satisfying the o.d.e. almost everywhere on $\mathbf{R}$. Prove or disprove:

Conjecture 2 *For $V \in L^2(\mathbf{R})$ there is a set of measure zero in $\mathbf{R}^+$ such that for $k \in \mathbf{R}^+$ not in this set all solutions to the above o.d.e are bounded (L^∞) functions.*

To trace some background information start with [2].

3 Tensor-Paraproduct

Let ϕ be a non-zero Schwarz function $\phi \in S(\mathbf{R})$, think of it as a smooth approximation to the characteristic function of $[-1, 1]$ Define

$$\phi_k(x) = 2^{-k}\phi(2^{-k}(x))$$

and the smoothing operators at scale 2^k:

$$P_k f(x) = f * \phi_k$$

This is a smooth version of the standard martingale average operator that averages on dyadic intervals of length 2^k, and the problem below is equally little understood in the case of the standard martingale operator. Define the difference operator $Q_k = P_k - P_{k-1}$.

Turning to functions in two variables x and y, define $P_{k,x}$ and $Q_{k,x}$ the corresponding operators acting only in the x variables, e.g.

$$P_k f(x, y) = \int f(x - t, y)\phi_k(t)\, dt$$

and similarly $P_{k,y}$ and $Q_{k,y}$.

Consider the bilinear operator acting on two functions f, g in two variables:

$$B(f, g) = \sum_{k \in Z} (P_{k,x} f)(Q_{k,y} g)$$

Problem: Prove or disprove

$$\|B(f, g)\|_{3/2} \leq C\|f\|_3\|g\|_3$$

4 0-1 degenerate bilinear Hilbert transform

This problem is somewhat similar to the previous tensor-paraproduct problem but most likely harder to prove a positive result and easier to find counterexample, as the case may be.

Define for two Schwartz functions f, g in $S(\mathbf{R}^2)$

$$B(f,g) = p.v. \int f(x+t,y)g(x,y+t)\frac{dt}{t}$$

Prove or disprove

$$\|B(f,g)\|_{3/2} \leq C\|f\|_3\|g\|_3$$

5 Two commuting transformations

Let X be a probability space and $S, T : X \to X$ two measure preserving transformations which commute: $ST = TS$. For $f, g \in L^\infty(X)$ define

$$A_n(x) = \frac{1}{N}\sum_{n=1}^{N} f(T^n x)g(S^n x)$$

Prove or disprove that for almost every x the sequence $A_n(x)$ is a convergent.

6 Sharp Beurling constant

The Beurling operator is the pricipal value convolution with $1/z^2$ in the complex plane. Problem: What is the exact norm of this operator in $L^p(\mathbf{R}^2)$.

There are several reformulations of this problem. Define $L(x)$ to be $|x| - |\overline{x}|^2$ for $|x| + |\overline{x}| < 1$ and $2|x| - 1$ otherwise. The prove $\int L(\nabla u) \geq 0$ for all $u \in W_0^{1/2}$. (See Baernstein, Montgomery-Smith: Some conjectures about integral means).

References

[1] Thiele, C. *Singular integrals meet modulation invariance* Proceedings ICM 2002 Beijing, Volume II

[2] Muscalu C., Tao T., Thiele C. *A Carleson type theorem for a Cantor group model of the scattering transform.* Preprint