

Singular integrals meet modulation invariance

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Abstract

Many concepts of Fourier analysis on Euclidean spaces rely on the specification of a frequency point. For example classical Littlewood Paley theory decomposes the spectrum of functions into annuli centered at the origin. In the presence of structures which are invariant under translation of the spectrum (modulation) these concepts need to be refined. This was first done by L. Carleson in his proof of almost everywhere convergence of Fourier series in 1966. The work of M. Lacey and the author in the 1990's on the bilinear Hilbert transform, a prototype of a modulation invariant singular integral, has revitalized the theme. It is now subject of active research which will be surveyed in the lecture. Most of the recent related work by the author is joint with C. Muscalu and T. Tao.

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1. Multilinear singular integrals

A basic example for the notion of *singular integral* is a convolution operator

$$Tf(x) = K * f(x) = \int K(x-y)f(y) dy \quad (1.1)$$

whose convolution kernel K is not absolutely integrable. If K was absolutely integrable then we had trivially an a priori estimate

$$\|K * f\|_p \leq \|K\|_1 \|f\|_p \quad (1.2)$$

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for $1 \leq p \leq \infty$. This follows by standard interpolation techniques from the two endpoints $p = 1, \infty$, which are true by trivial manipulations.

A basic point of singular integral theory is that an estimate of the form (1.2) may prevail for $1 < p < \infty$ with a constant $C_{p,K}$ instead of $\|K\|_1$ on the right hand side, if K is not absolutely integrable and the integral (1.1) is only defined in a distributional (principal value) sense. The most prominent example on the real line (indeed, all operators in this article will act on functions on the real line) is the Hilbert transform with $K(x) = 1/x$.

Taking formally Fourier transforms, one can write (1.1) as multiplier operator:

$$\widehat{Tf}(\xi) = \widehat{K}(\xi)\widehat{f}(\xi) =: m(\xi)\widehat{f}(\xi) \quad . \quad (1.3)$$

For the purpose of this survey a sufficiently interesting class of singular integrals is described in terms of the multiplier m by imposing the symbol estimates

$$(d/d\xi)^\alpha m(\xi) \leq C|\xi|^{-\alpha} \quad (1.4)$$

for $\alpha = 0, 1, 2$. We define the dual bilinear form

$$\Lambda(f_1, f_2) = \int (Tf_1(x))f_2(x) dx = \int_{\xi_1+\xi_2=0} \widehat{f}_1(\xi_1)\widehat{f}_2(\xi_2)m(\xi_1) d\sigma \quad (1.5)$$

where $d\sigma$ is the properly normalized Lebesgue measure on the hyperplane $\xi_1 + \xi_2 = 0$. The natural generalization of estimate (1.2) using duality of L^p spaces then takes the form

$$|\Lambda(f_1, f_2)| \leq C_{p_1} \|f_1\|_{p_1} \|f_2\|_{p_2} \quad (1.6)$$

with $1/p_1 + 1/p_2 = 1$.

Estimate (1.6) can be related to square function estimates which are fundamental in singular integral theory. Let $(\psi_j)_{j \in \mathbf{Z}}$ be a family of functions such that $m_j := \widehat{\psi_j}$ is supported in the ball $B(0, 2^j)$ of radius 2^j around 0, vanishes on $B(0, 2^{j-2})$, and satisfies the symbol estimates (1.4) uniformly in j . By square function estimate we mean the inequality

$$\|(\sum_j |f * \psi_j|^2)^{1/2}\|_p \leq C_p \|f\|_p \quad (1.7)$$

which holds for $1 < p < \infty$. Now let m be any multiplier satisfying (1.4). It is easy to split it as $m(\xi_1) = \sum_j \widehat{\psi}_{1,j}(\xi_1)\widehat{\psi}_{2,j}(-\xi_1)$ for two families $\psi_{1,j}$ and $\psi_{2,j}$ as in the square function estimate. Then we have

$$|\Lambda(f_1, f_2)| = \left| \sum_j \int (f_1 * \psi_{1,j})(x)(f_2 * \psi_{2,j})(x) dx \right| \quad .$$

Moving the sum inside the integral and applying Cauchy-Schwarz, Hölder, and (1.7) we obtain (1.6).

A natural generalization (see [14]) of (1.5) to multilinear forms is

$$\Lambda(f_1, \dots, f_n) = \int_{\xi_1 + \dots + \xi_n = 0} m(\xi_1, \dots, \xi_{n-1}) \prod_{j=1}^n \widehat{f}_j(\xi_j) d\sigma \quad (1.8)$$

with multipliers m satisfying

$$\partial^\alpha m(\xi') \leq C |\xi'|^{-|\alpha|} . \quad (1.9)$$

Here $\xi' = (\xi_1, \dots, \xi_{n-1})$ and α runs through all multi-indices up to some order N . Note that the special role of the index n in the above is purely notational. The natural estimates to ask for are

$$|\Lambda(f_1, \dots, f_n)| \leq C_{p_1, \dots, p_{n-1}} \prod_{j=1}^n \|f_j\|_{p_j} \quad (1.10)$$

for $1 < p_j < \infty$ with $\sum_j 1/p_j = 1$. In the special case that m is constant, $\Lambda(f_1, \dots, f_n)$ is a multiple of the integral of the pointwise product of the functions f_j and estimate (1.10) is simply Hölder's inequality.

We sketch a proof of (1.10). Without destroying the symbol estimates, we can split m into a finite sum of multipliers, each supported on a narrow cone with tip at the origin. Thus assume m is supported on such a cone consisting of rays having small angle with a vector η' .

We may assume by symmetry that $\eta'_1 = 1$ is the maximal component of η' . Then we can split m into pieces m_j satisfying (1.9) uniformly and supported in

$$(B(0, 2^j) \setminus B(0, 2^{j-2})) \times B(0, 2^{j+n})^{n-2} .$$

Introduce η_n such that $\sum_j \eta_j = 0$. By symmetry among the indices larger than 1 we may assume $\eta_2 \geq 1/n$. Then it is easy to arrange (see Figure "Cone") the support of m_j to be in

$$(B(0, 2^j) \setminus B(0, 2^{j-2})) \times (B(0, 2^{j+n}) \setminus B(0, 2^{j-n})) \times B(0, 2^{j+n})^{n-3} .$$

Using smoothness of the multiplier m_j we may use Fourier expansion to write it as rapidly converging sum of multipliers of elementary tensor form

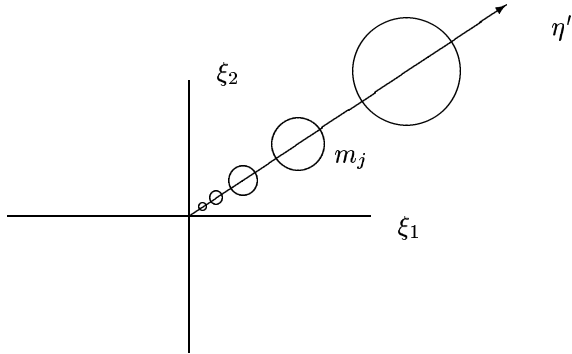
$$\widehat{\psi}_{1,j}(\xi_1) \widehat{\psi}_{2,j}(\xi_2) \prod_{l=3}^n \widehat{\phi}_{l,j}(\xi_l)$$

with $\xi_n = -\sum_{j=1}^{n-1} \xi_{n-1}$. The symbol estimates prevail for these elementary tensors, and thus we observe

$$(d/d\xi)^\alpha (\phi_{l,j})(\xi) \leq C 2^{-\alpha j} \quad (1.11)$$

for all derivatives up to order N . Observe that $\widehat{\psi}_{l,j}$ are essentially as in (1.7), and $\widehat{\phi}_{l,j}$ are similar but fail to be supported away from the origin. Applying the

Figure 1: “Cone”



elementary tensor multiplier form to f_1, \dots, f_2 is the same as applying a constant multiplier to $\psi_{1,j} * f_1, \dots, \phi_{n,j} * f_n$. Estimate (1.10) then follows from

$$\begin{aligned} & \sum_j \int \prod_{l=1}^2 (\psi_{l,j} * f_l)(x) \prod_{l=3}^n \phi_{l,j} * f_l(x) d\sigma \\ & \leq C \prod_{l=1}^2 \left\| \left(\sum_j |f_l * \psi_{l,j}|^2 \right)^{1/2} \right\|_{p_l} \prod_{l=3}^{\infty} \left\| \sup_j \|f_l * \phi_{l,j}\|_{p_l} \right\| \leq C \prod_{l=1}^n \|f_l\|_{L_{l_i}^p} . \end{aligned}$$

Here we have used for $l = 1, 2$ the square function estimate (1.7) and for $l > 2$ the equally fundamental Hardy Littlewood maximal inequality

$$\left\| \sup_j |f * \phi_{l,j}| \right\|_{L^p} \leq C_p \|f\|_p$$

which is valid due to (1.11).

2. Modulation invariance

Modulation M_η with parameter $\eta \in \mathbb{R}$ is defined to be multiplication by a character:

$$M_\eta f(x) := f(x) e^{2\pi i \eta x} .$$

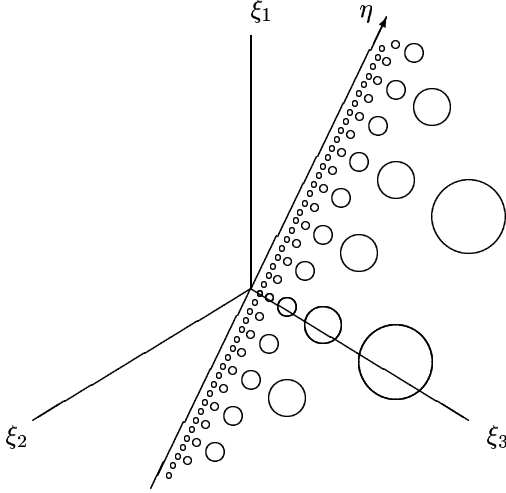
This amounts to a translation of the Fourier transform of f .

We shall be interested in multilinear forms Λ which have modulation symmetries in the sense

$$\Lambda(f_1, \dots, f_n) = \Lambda(M_{\eta_1} f_1, \dots, M_{\eta_n} f_n) \quad (2.12)$$

for all vectors $\eta = (\eta_1, \dots, \eta_n)$ in a subspace Γ of the hyperplane given by $\sum \eta_j = 0$.

Figure 2: “Circles”



If Λ is given in multiplier form (1.8), then (2.12) is equivalent to a translation symmetry of the multiplier m :

$$m(\xi_1, \dots, \xi_n) = m(\xi_1 + \eta_1, \dots, \xi_n + \eta_n) . \quad (2.13)$$

Such a symmetry with nontrivial η is inconsistent with the symbol estimates (1.9) unless m is constant. Namely, by iterating (2.13), any point with nonvanishing derivative of m can be translated to a point far away from the origin, until the value of the derivative, which remains constant at the translated points, contradicts (1.9).

A natural replacement for (1.9) in the presence of modulation symmetry along vectors in Γ has been introduced by Gilbert/Nahmod [6]:

$$\partial^\alpha m(\xi') \leq C \text{dist}(\xi', \Gamma')^{-|\alpha|} . \quad (2.14)$$

Here Γ' is the projection of Γ onto the first $n - 1$ coordinates. Figure “Circles” indicates the regions in which multipliers of the form (2.14) can be thought of as being essentially constant.

The following theorem is due to [6] in the case $n = 3$ and to [16] in general:

Theorem 2..1 *Assume $k := \dim(\Gamma) < n/2$, and assume that Γ is non-degenerate in the sense that for any $1 \leq i_1 < \dots < i_k \leq n$ the space Γ is the graph of a function in the variables $\xi_{i_1}, \dots, \xi_{i_k}$. Assume m satisfies (2.14). Then Λ as in (1.8) satisfies (1.10) whenever $\sum 1/p_j = 1$ and $1 < p_j \leq \infty$ for all p_j .*

We remark that it is unknown whether the condition $\dim(\Gamma) < n/2$ can be relaxed in this theorem.

The forms Λ have dual multilinear operators. Theorem 2..1 implies a priori estimates for these multilinear operators. Moreover, these multilinear operators satisfy estimates which cannot be formulated in terms of L^p estimates for Λ . Let (p_1, \dots, p_n) be a tuple of real numbers or ∞ such that at most one of these numbers is negative. If all of them are nonnegative, we say Λ is of type (p_1, \dots, p_n) if (1.10) holds. If one of them, say p_j , is negative, then we define the dual operator T by

$$\Lambda(f_1, \dots, f_n) = \int T(f_1, \dots, f_{j-1}, f_{j+1}, \dots, f_n)(x) f_j(x) dx \quad .$$

We then say that Λ is of type (p_1, \dots, p_n) if

$$\|T(f_1, \dots, f_{j-1}, f_{j+1}, \dots, f_n)\|_{p'_j} \leq C \prod_{i \neq j} \|f_i\|_{p_i}$$

where $p'_j = p_j/(p_j - 1)$. Observe $0 < p'_j < 1$. The following theorem is again due to [6] ($n = 3$) and [16]:

Theorem 2..2 *Let Γ and Λ be as in Theorem 2..1. Then Λ is of type (p_1, \dots, p_n) if $\sum_j 1/p_j = 1$, at most one of the p_j is negative, none of the p_j is in $[0, 1]$, and*

$$1/p_{i_1} + \dots + 1/p_{i_r} < \frac{n - 2\dim(\Gamma) + r}{2}$$

for all $1 \leq i_1 < \dots < i_r \leq n$ and $1 \leq r \leq n$.

A basic example of a modulation invariant form Λ is when $n = 3$ and $m(\xi_1, \xi_2)$ is constant on both sides of a line Γ but not globally constant. With proper choice of constants this form can be written as

$$\Lambda_\alpha(f_1, f_2, f_3) = \int B_\alpha(f_1, f_2)(x) f_3(x) dx$$

with the bilinear Hilbert transform

$$B_\alpha = p.v. \int f_1(x - t) f_2(x - \alpha t) \frac{1}{t} dt$$

and a (projective) parameter α determining the direction of the line Γ . Theorems 2..1 and 2..2 in this special case are due to [10] and [11].

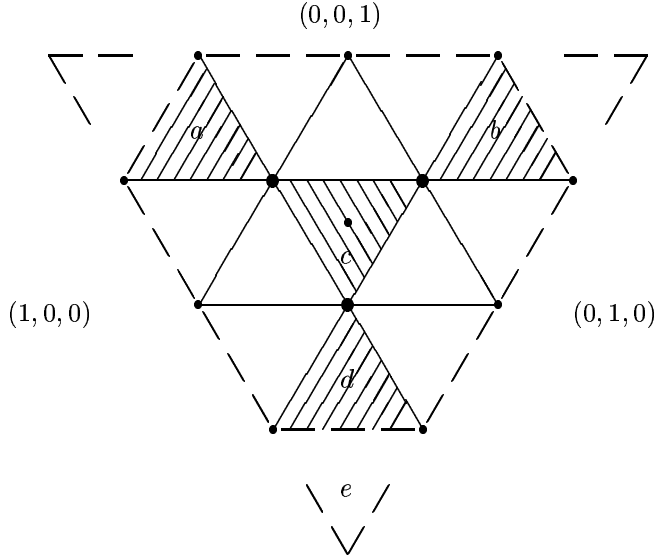
For the bilinear Hilbert transform nondegeneracy specializes to the condition $\alpha \notin \{0, 1, \infty\}$, and the conclusion of both theorems can be summarized to

$$\|B_\alpha(f_1, f_2)\|_p \leq C_{p_1, p_2} \|f_1\|_{p_1} \|f_2\|_{p_2} \quad (2.15)$$

provided $1 < p_1, p_2 \leq \infty$ and $2/3 < p < \infty$. The set of types of such Λ_α is the convex hull of the open triangles a, b, d in Figure ‘‘Hexagon’’ which depicts the plane of $(1/p_1, 1/p_2, 1/p_3)$ with $\sum_j 1/p_j = 1$. It is unknown whether the type-region of Λ_α extends to the open triangle e and its symmetric counterparts.

We point out a related result by M. Lacey [9]:

Figure 3: “Hexagon”



Theorem 2.3 *The maximal truncations of the bilinear Hilbert transform,*

$$B_{\alpha}^{\max}(f, g)(x) := \sup_{\epsilon > 0} \left| \int_{\mathbb{R} \setminus [-\epsilon, \epsilon]} f(x-t)g(x-\alpha t) \frac{1}{t} dt \right|$$

also satisfy (2.15) provided α is not degenerate.

This is stronger than the bounds for the bilinear Hilbert transform itself.

The main difference in proving the theorems in this section compared to the discussion in Section 1. is that it is not sufficient to split the functions f_k into frequency parts supported in $B(0, 2^j) \setminus B(0, 2^{j-2})$. The special role that is attributed to the zero frequency by this splitting is obsolete in the modulation invariant setting. Instead one has to consider frequency bands of f_k away from the origin and very narrow, such as intervals $[N - \epsilon, N + \epsilon]$ for large N and small ϵ . Geometrically these bands can be viewed as the projections of the circles in Figure “Circles” onto the projected coordinate axes. Handling thin frequency bands requires a new set of techniques. Prior to the work [10] and [11] these techniques have been pioneered in [2] and [5] where the Carleson operator

$$Cf(x) = \sup_{\xi} |p.v. \int e^{iy\xi} f(x-y) \frac{1}{y} dy|$$

has been estimated. Note that this operator is modulation invariant, $C(f) =$

$C(M_\eta f)$. See also [12]. Most theorems discussed in this survey have a simpler but significant model theorem in the dyadic setting, see for example [17], [22],

3. Uniform estimates

Theorem 2.1 excludes certain degenerate subspaces Γ . For some degenerate Γ the multilinear forms split into simpler objects and one can provide L^p estimates also in these degenerate cases; we will give examples below. This raises the question whether one can prove bounds on Λ uniformly in the choice of Γ , as Γ approaches one of these degenerate cases.

Substantial progress on this question has only been made in the case $\dim(\Gamma) = 1$.

Theorem 3.1 *Let $n \geq 3$ and (η_1, \dots, η_n) be a unit vector spanning the space Γ , and assume $\eta_j \neq 0$ for all j . Define the metric*

$$d(x, y) := \sup_{1 \leq j \leq n} \frac{|x_j - y_j|}{|\eta_j|}$$

and write $d(x, \Gamma) := \inf_{y \in \Gamma} d(x, y)$. Suppose m satisfies the estimate

$$\partial_{\eta'}^\alpha m(\eta') \leq \prod_{j=1}^n (\eta_j d(\eta, \Gamma))^{-\alpha_j} \quad (3.16)$$

for all partial derivatives $\partial_{\eta'}^\alpha$ up to order N . Then (1.10) holds for all $2 < p_j < \infty$ with $\sum_j 1/p_j = 1$ with the bounds uniform in the choice of Γ .

We discuss uniform estimates for the special case of the bilinear Hilbert transform. The degenerate directions for Γ occur when the vector η is perpendicular to one of the three projected coordinate axes (see Figure ‘‘Circles’’). One of the degenerate cases ($\alpha = 1$) gives rise to the operator

$$B_1(f_1, f_2) = H(f_1 \cdot f_2)$$

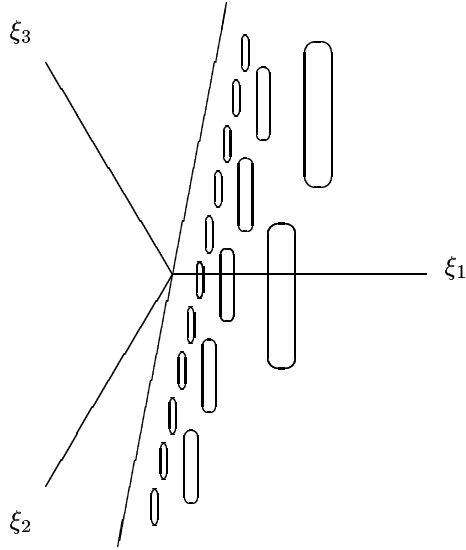
(Hilbert transform of the pointwise product) or its dual operators

$$f_2 \cdot H(f_3), \quad f_1 \cdot H(f_3) \quad .$$

Besides the usual homogeneity $\sum_j 1/p_j = 1$, the only constraint for these operators to be of type (p_1, p_2, p_3) is $1 < p_3 < \infty$. In Figure ‘‘Hexagon’’ this region is the strip bounded by the horizontal lines through $(0, 0, 1)$ and $(1, 0, 0)$.

Thus one expects the constants in the L^p estimates to be uniform as α approaches 1 in the intersection of this strip and the convex hull of triangles a, b, d . The above theorem provides uniform estimates in the inner triangle c . This special case of Theorem 3.1 was previously shown by Grafakos/Li [7], and Li [13] has shown uniform estimates in triangles a and b . These results together give uniform bounds in the convex hull of a, b, c . Uniform estimates near the points $(1, 0, 0)$ and

Figure 4: “Ellipses”



$(0, 1, 0)$ remain an open question. Prior to the work of Grafakos/Li [7], weak type uniform bounds were shown [23], [24] in the common boundary point of triangles a and c (and by symmetry also b and c).

The multiplier condition (3.16) gives essentially constant multipliers on regions adapted to the slope of Γ , see Figure “Ellipses”. Observe that all ellipses at a given scale project essentially onto disjoint regions when projected to any one of the coordinate axes. Handling these adapted regions uniformly requires considerable refinements of the arguments in [10] and [11].

We mention that closely related to the topic of uniform estimates for the bilinear Hilbert transform is that of bilinear multiplier estimates for multipliers which are singular along a curve rather than a line, provided the curve is tangent to a degenerate direction. Results for such multipliers have been found by Muscalu [15] and Grafakos/Li [8].

We conclude this section with a remark on the history of the bilinear Hilbert transform. Calderon is said to have considered the bilinear Hilbert transform in the 1960's while studying what has been named Calderon's first commutator. This is the bilinear operator

$$\mathcal{C}(A, f)(x) = p.v. \int \frac{A(x) - A(y)}{(x - y)^2} f(y) dy \quad .$$

It can be viewed as a bilinear operator in the derivative A' of A and the function f , and as such has a multiplier form as in (1.8). To see this, we can write $C(A, f)$ in terms of A' as a superposition of bilinear Hilbert transforms:

$$\begin{aligned} \mathcal{C}(A, f)(x) &= p.v. \int \int_0^1 A'(x + \alpha(y - x)) \frac{1}{x - y} f(y) d\alpha dy \\ &= \int_0^1 B_\alpha(f, A')(x) d\alpha \quad . \end{aligned}$$

The estimate Calderon was looking for was

$$\|C(A, f)\|_2 \leq \|A'\|_\infty \|f\|_2 \quad . \quad (3.17)$$

Thus he needed good control over the constant C_α as α approaches 0 or 1. However, even finiteness of C_α was not known to Calderon. Sufficiently good control over C_α was first established in [23].

The multiplier of $\mathcal{C}(A', f)$ is more regular than that of the bilinear Hilbert transform, and Calderon, quitting his attempts to estimate the bilinear Hilbert transform, proved estimate (3.17) by refinements of the methods in Section 1. (see [1]).

4. More multilinear operators

Theorem 2..1 discusses multipliers singular at a single subspace Γ' . Cut and paste arguments easily allow to generalize the theorem to the case of multipliers singular at finitely many subspaces $\Gamma_1', \dots, \Gamma_k'$, provided each subspace satisfies the dimension and non-degeneracy conditions of Theorem 2..1.

Interesting phenomena occur for multipliers singular at several subspaces $\Gamma_1', \dots, \Gamma_k'$ which do not satisfy the conditions of Theorem 2..1. Some operators corresponding to multipliers singular at degenerate subspaces can be written in terms of pointwise products and lower degree operators and thus can be trivially shown to satisfy L^p estimates. If m is singular at several such subspaces, the trivial splitting may no longer be possible, and one has to do a much more subtle analysis.

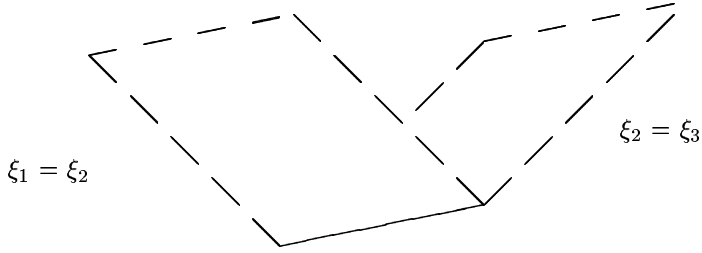
We consider the special case when the spaces $\Gamma_1', \dots, \Gamma_k'$ are hyperplanes and the multiplier is the characteristic function of one of the infinite simplices been cut out of \mathbb{R}^n by these hyperplanes, see Figure “Wedge”. A basic example is the trilinear operator

$$T(f_1, f_2, f_3)(x) = \int_{\alpha_1 \xi_1 < \alpha_2 \xi_2 < \alpha_3 \xi_3} \prod_{j=1}^3 \widehat{f}_j(\xi_j) e^{2\pi i x \xi_j} d\xi_j$$

and its associated fourlinear form

$$\Lambda(f_1, f_2, f_3, f_4) = \int_{\sum_{j=1}^4 \xi_j = 0, \alpha_1 \xi_1 < \alpha_2 \xi_2 < \alpha_3 \xi_3} \prod_{j=1}^4 \widehat{f}_j(\xi_j) d\sigma \quad . \quad (4.18)$$

Figure 5: “Wedge”



Here $\alpha_1, \alpha_2, \alpha_3$ are real parameters. If we had only one of the two constraints $\alpha_1 \xi_1 < \alpha_2 \xi_2$ or $\alpha_2 \xi_2 < \alpha_3 \xi_3$, then these operators would decompose trivially.

There is a Zariski open set of values of $(\alpha_1, \alpha_2, \alpha_3)$ for which Λ and T are well behaved. The following theorem proved in [18] states such estimates for the generic point $(1, 1, 1)$.

Theorem 4..1 *For $\alpha_1, \alpha_2, \alpha_3 = 1$ the form Λ as in (4.18) satisfies estimates*

$$\Lambda(f_1, f_2, f_3, f_4) \leq C_{p_1, \dots, p_4} \prod_{j=1}^4 \|f_j\|_{p_j}$$

if $1 < p_j < \infty$ and $\sum_j 1/p_j = 1$. The trilinear form T satisfies in addition estimates mapping into L^p with $p < 1$, in particular

$$\|T(f_1, f_2, f_3)\|_{2/3} \leq C \prod_{j=1}^3 \|f_j\|_2 .$$

An example for a degenerate choice of $(\alpha_1, \alpha_2, \alpha_3)$ is $(1, -1, 1)$. In this case there is a negative result [19]:

Theorem 4..2 *For $\alpha_1 = 1, \alpha_2 = -1, \alpha_3 = 1$ the a priori estimate*

$$\|T(f_1, f_2, f_3)\|_{2/3} \leq C \prod_{j=1}^3 \|f_j\|_2$$

does not hold.

Theorem 4.2 is proved by applying T to functions f_1, f_2, f_3 which are suitable truncations of imaginary Gaussians (chirps) $e^{i\beta x^2}$. The operator of Theorem 4.2 appears naturally in eigenfunction expansions of one dimensional Schrödinger operators, see the work of Christ/Kiselev [3],[4]. A positive result on discrete models of these expansions using the modulation invariant theory can be found in [20].

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