1. Lecture 5

1.1. **Walsh phase plane.** The Phase plane is a cartoon for $L^2(\mathbb{R})$. We use it to visualize selected functions in $L^2(\mathbb{R})$, and interpret Hilbert space properties such as subspaces and inner products in this cartoon.

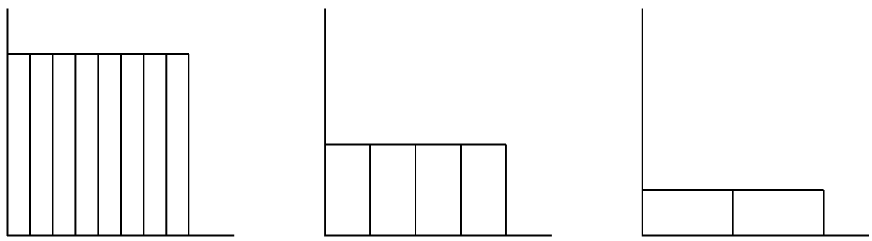
We shall restrict attention to the positive real line, note that dyadic intervals live either entirely in the negative of positive half line and the two half lines are therefore completely separated.

The dyadic phase lane is then the first quadrant in the plane. The horizontal axis corresponds to the usual x-axis representing the argument of a function, while the second coordinate is some dual space that will be clarified momentarily.

We first represent the L^2 normalized characteristic function

$$\chi_I = |I|^{-1/2} 1_I$$

of a dyadic interval as a rectangle of area 1 such that one edge is the interval I on the real line The following figure shows such rectangles at three different scales.



We make the immediate observation, that two characteristic functions are orthogonal to each other, if and only if their rectangles are disjoint. To be precise we continue to assume all intervals are left closed and right open, so the rectangles contain the boundary to bottom and left but not the boundary to upper and right.

Moreover, the inner product of two characteristic functions is equal to the square root of the area of intersection of the rectangles, namely, if nonzero. Namely, if $I \subset I'$, then

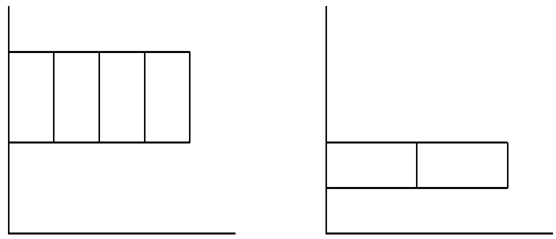
$$\langle \chi_I, \chi_{I'} \rangle = \int_I |I|^{-1/2} |I'|^{-1/2} \, dx = \sqrt{|I_1|/|I_2|} \ .$$

This property already motivates the choice of the height of the rectangles.

The Haar functions on an interval I,

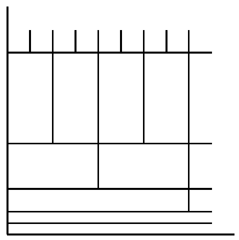
$$h_I(x) = |I|^{-1/2} (1_{I_l} - 1_{I_r})$$

is represented by a rectangle that is the translate of the rectangle for the characteristic function by one side-length up in the plane.



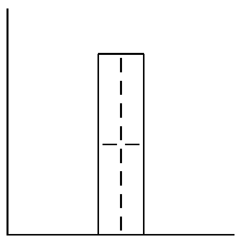
Note that still we have the property that the inner product of two function is equal to the square root of the area of intersection of the rectangles. In particular all Haar functions correspond to pairwise disjoint rectangles.

The union of rectangles of all Haar functions covers the entire phase plane



This is in good relation with the fact that the Haar functions form a complete orthonormal basis of $L^2(\mathbb{R}^+)$.

Now consider a rectangle of area two whose bottom edge is a dyadic interval on the horizontal axis. It can be split into rectangles of area one in two ways: left and right sub-rectangle upper and lower sub-rectangle.

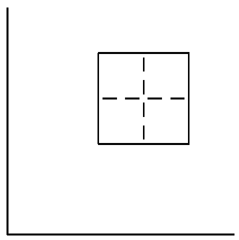


All these rectangles correspond to functions, and we have the properties

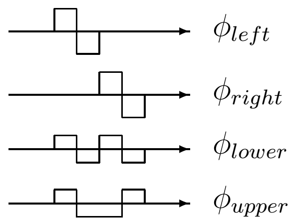
$$\begin{aligned} \phi_{lower} &:= (\phi_{left} + \phi_{right})/\sqrt{2} \\ \phi_{upper} &:= (\phi_{left} - \phi_{right})/\sqrt{2}\ . \end{aligned}$$

Note that this is an orthonormal transformation from the pair $(\phi_{left}, \phi_{right})$ to the pair $(\phi_{left}, \phi_{right})$. In particular both pairs are an orthonormal basis of the same subspace of $L^2(\mathbb{R})$ that we may consider represented by the rectangle of area 2.

It is natural to generalize this construction to other dyadic rectangles of area 2, for example the union of two Haar functions.



The upper and lower tile do not yet correspond to any functions we have already defined. We use the above orthogonal transformation to define such functions. The functions involved are as follows:



We now make a formal definition:

Definition 1.1. *A tile is a dyadic rectangle of area one in the first quadrant, i.e.;*

$$p := I \times \omega := [2^k n, 2^k(n+1)) \times [2^{-k} l, 2^{-k}(+1))$$

for integers k and $n \geq 0, l \geq 0$. The integer l is called the height of p.

If $l = 0$, then we define the Walsh wave packet ϕ_p to be the normalized characteristic function of the interval I:

$$\phi_p := |I|^{-1/2} 1_I\ .$$

If $l > 0$, we assume we have already defined Walsh wave packets for all tiles with height less than l.

If l is even, then p is the lower tile of the union of a left tile p_l and a right tile p_r with parameters

$$(k-1, 2n, l/2),\ (k-1, 2n+1, l/2)$$

and we define

$$\phi_p := 2^{-1/2}(\phi_{p_l} + \phi_{p_r})\ .$$

If l is odd, then p is the upper tile of the union of a left tile p_l and a right tile p_r with parameters

$$(k-1, 2n, (l-1)/2),\ (k-1, 2n+1, (l-1)/2)$$

and we define

$$\phi_p := 2^{-1/2}(\phi_{p_l} - \phi_{p_r})\ .$$

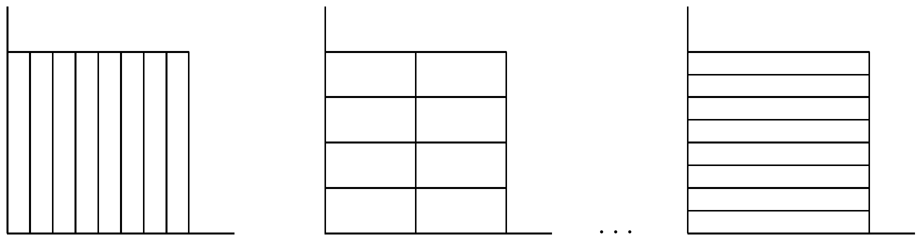
By induction it is very easy to verify the following simple properties about the shape of the functions ϕ_P.

Proposition 1.1.

(1) *ϕ_p is supported on I.*
(2) *ϕ_p has constant modulus $2^{-k/2}$ on I.*
(3) *If p and p' have the same height, then there is an affine transformation L of the real line, namely the orientation preserving transformation that maps I to I', such that $\phi_p(x) = \phi_{p'}(L^{-1}x)$.*
(4) *For each integer $l' \geq 0$ there is a unique height l such that the Walsh wave packet of each tile p with height l has l' points of discontinuity in the interior of I. I for some integer K we have $2^K \leq l' < 2^{K+1}$ then also $2^K \leq l < 2^{K+1}$.*

Unfortunately, in general we do not have $l = l'$ in property (4) as we can already see from $l = 2, 3$ whose wave packets are pictured above One could reorganize the theory to have $l = l'$ in (4), at the expense of having a more complicated choice of sign in the recursion rules for ϕ_p. We prefer the simple recursion rule. Note that the functions $f_l(x) = \sin(\pi l x)$ satisfy a similar property as (4) on the interval $[0, 1)$ if the word discontinuity is replaced by zero crossing.

Consider the space of functions on $[0, 1)$ which are constant on intervals of length 2^k, k a negative number of large absolute value. This space has as orthonormal basis the set of normalized characteristic functions of dyadic intervals of that scale inside $[0, 1)$. Applying the above orthogonal transformation on each pair of dyadically adjacent intervals, we obtain another basis for that space, consisting of normalized characteristic functions and Haar functions. Iterating this we can generate a sequence of basis corresponding to uniform tilings of the same area. The last basis consists of wave packets on the interval $[0,1)$ itself. They are parametrized by the height l and called the Walsh-Fourier basis functions w_l on $[0,1)$



Increasing the parameter k we see that the definition of w_l for fixed l does not change, only the collection of Walsh functions increases. Taking a limit as $k \to \infty$, we see that $\{w_l\}_{l \in \mathbb{N}_0}$ is an orthonormal basis of $L^2(\mathbb{R})$. Namely, they span the same space as the set of Haar functions on $[0,1)$. This Walsh Fourier basis is very reminiscent of the Fourier basis of $L^2(\mathbb{R})$.

Define the Rademacher function r_k to be

$$r_k = \sum_{|I|=2^{-k}} |I|^{1/2} h_I$$

the normalization being such that r_k has constant modulus one. Rademacher functions have the property that haar coefficients at the same scale are equal, hence any linear combination of Rademacher functions satisfies Khintchine's inequality. We discuss the effect of multiplying a Walsh wave packet ϕ_p by a Rademacher functions r_k. If $|I| < 2^k$ then r_k is constant on the support of ϕ_p and the effect is the same as multiplying ϕ_p by 1 or -1. This does not change the space spanned by ϕ_p, in particular leaves the projection

$$\langle f, \phi_p \rangle \, \phi_p$$

invariant. If $|I| = 2^k$, then the effect is on of interchanging the sign in the orthogonal transformation defining ϕ_p. Hence it interchanges the wave packets of vertical siblings at scale 2^k. We may define a measure preserving interval exchange transformation of the horizontal axis, interchanging dyadic siblings at scale 2^{-k}. This leads to a natural transformation of the Walsh phase plane, acting identically on each fiber for fixed x. By induction on the recursive definition of ϕ_p it is clear that for $|I| \geq 2^k$ we have

$$r_k \phi_p = \phi_{p'}$$

where p' is the image of p under this transformation. For $|I| \leq 2^k$ we had observed

$$r_k \phi_p = \pm \phi_p = \pm \phi_{p'}$$

since $p = p'$ in this case. We note that the interval exchange transformation preserves dyadic distance. Here dyadic distance of two numbers is defined as the length of the shortest dyadic interval containing both numbers.

Consider a Walsh function w_l on the interval $[0, 1)$. Unless $l = 0$, let k be the smallest integer such that $l < 2^k$, and consider

$$r_k w_l \ .$$

By the above discussion this is again a Walsh wave packet $w_{l'}$ and we have $l' < 2^{k-1}$ since l by definition is in the upper child of $[0, 2^l)$ so l' is in the lower child. Iterating this procedure, we eventually arrive at the Walsh function w_0. Noting that $r_k^2 = 1$ for each k, we therefore have written w_l as product of Rademacher functions on the interval $[0, 1)$ (empty product if $l = 0$). Since there are 2^k different Walsh functions which can be written as product of the first k Rademacher functions, we see that each of the possible 2^k products of these Rademacher functions occurs as a Walsh function.

Hence the Walsh functions are the multiplicative group of functions on $[0, 1)$ generated by the Rademacher functions. (In abstract Fourier analysis one can identify this group as group of characters of an underlying group structure on $[0, 1)$. More precisely, the underlying group is a set that is Lebesgue-equivalent to $[0, 1)$, that is to a set of measure zero. We can - up to a set of measure zero - identify $[0, 1)$ with $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}_0}$ by writing almost every number in $[0, 1)$ uniquely in binary expansion. The group structure on $\mathbb{Z}/2\mathbb{Z}^{\mathbb{N}_0}$, namely bit-wise addition without carry over, leads to the Walsh functions as characters.)

We have the following generalization of an observation made earlier:

Proposition 1.2. *For any two tiles p, p' we have*

$$|\langle \phi_p, \phi_{p'} \rangle| = |p \cap p'|^{1/2} \ .$$

In particular two wave packets are perpendicular if and only if the underlying tiles have empty intersection.

Proof: There is a dyadic interval $[0, 2^N)$ for sufficiently large N which contains I and I'. By scaling we may assume $N = 0$. If $|I| = |I'|$ then both p and p' are members of a uniform tiling discussed above, hence they are either different and perpendicular or equal as claimed. If $|I| > |I'|$, we do induction on the difference in scales. Thus decompose I into its dyadic children $I_l \cup I_r$ and note that I' is either contained in

I_l or I_r or disjoint from both. In either of the cases the claim follows easily by induction, writing ϕ_P as linear combination of left and right tile. $\square$

Proposition 1.3. *Suppose q is a tile and $\mathbf{p}$ a finite collection of tiles whose union contains $\mathbf{p}$. Then ϕ_q is in the linear span on $\{\phi_p\}_{p \in \mathbf{p}}$*

We prove this by induction on the cardinality of $\mathbf{p}$. If $\mathbf{p}$ has only one element p, then necessarily $p = q$ and the statement is trivial.

Now assume that the cardinality of $\mathbf{p}$ is larger than 1. We may assume that no proper subset of $\mathbf{p}$ covers q. Let p be a tile in $\mathbf{p}$ with maximal scale I. Clearly it does not have the same scale as q or else it is equal q. Assume it has larger scale. Let $\tilde{p}$ be the vertical sibling of p. Note that $\tilde{p} \cap q \subset q$ since $I_q \subset I_p$ and $\omega_{\tilde{p}} \subset \omega_q$ by properties of dyadic intervals. No point in $\tilde{p} \cap q$ can be covered by a tile in P with larger spatial interval than $\tilde{p}$, and not all points in $\tilde{p} \cap p'$ can be covered by a tile in P with smaller spatial interval than $\tilde{p}$ because then p could be removed from $\mathbf{p}$ and the remaining set would still cover q. Hence $\tilde{p} \in \mathbf{p}$.

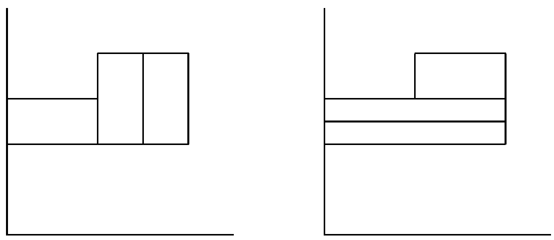
But then $p, \tilde{p}$ cover a bitile, and we can replace this pair by the horizontal siblings in the bitile without changing the span of the collection of functions. Only one of the siblings $\tilde{\tilde{p}}$ intersects q, but then $(\mathbf{p} \setminus \{p, \tilde{p}\}) \cup \{\tilde{\tilde{p}}\}$ covers q and by induction ϕ_q is contained in the span of the corresponding wave packets.

If the scale of p is less than that of q, one can do a similar geometric argument with roles of I and ω interchanged.

This proves the proposition. $\square$

Corollary 1.2. *Suppose we have two finite collection $\mathbf{p}$ and $\mathbf{p}'$ of tiles, and suppose both partition the same set in the phase plane. Then both form an orthonormal basis of the same subspace of $L^2(\mathbb{R})$.*

Note that the union of these intervals can therefore be uniquely associated with a subspace of $L^{(}\mathbb{R}^+)$. The following shows an example of



two such collections of tiles.

Proof: Both collections of wave packets are orthonormal sets, and by the previous proposition, they have the same span.

□