Bootcamp

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0.1 Compactness

Definition 1 *A metric space is called compact, if every cover of the space has a finite subcover.*

As in the case of separability we have the following two observations:

Lemma 1 *Finite sets are compact.*

Lemma 2 *If a metric subspace M' of a space M is compact and we are given any collection of balls in M that cover M', then there is a finite subcover.*

The analogy of the theories of separable and compact sets has an end when the countability of the set of radii $\mathbf{D}$ comes into play. For example, compact spaces do not have finite dense sets (finite sets are never dense in any larger space). Also, a subset of a compact set need not be compact.

On the other hand one has some immediate consequences from applying the compactness property to all balls centered at a given point:

Definition 2 *A nonempty metric space (M, d) is bounded if it is empty or there exists a ball which contains the whole space M.*

This is clear if M is empty by using the empty ball. Note that if K is a nonempty subset of another metric space M, then it suffices to find a ball in M that covers K. Namely, if $B_r(y)$ covers K and $x \in K$, then $B_{2r}(x)$ also covers K by the triangle inequality.

Lemma 3 *A compact metric space (K, d) is bounded.*

Proof: We may assume K is not empty. Pick a point $x \in K$ and observe that

$$M = \bigcup_{r \in \mathbf{D}} B_r(x)$$

, since every point y in K has some finite distance $d(x, y)$ to x and there exists $r \in \mathbf{D}$ with $d(x, y) < r$ and hence $y \in B_t(x)$. Hence the collection of balls about x covers

the compact set, and hence there is a finite collection of balls about x which cover the compact set. Since the balls are nested, it suffices to take the largest ball to cover the compact set. This proves boundedness. $\square$

There is a dual observation using complements of balls

Lemma 4 *Every compact metric space (K, d) is complete.*

Note that the notion of completeness is also independent of the embedding.

Proof:

Pick a cone B in K. For every $(x, r) \notin B$ there exists $\epsilon > 0$ and $(y, s) \in B$ such that

$$d(x, y) \geq r + s + \epsilon$$

Pick the ball $B_\epsilon(r)$ and let C be the collection of all these balls.

We have two cases:

1. Assume C does cover K. Then there is some finite subcover. Each ball in the cover comes with apoint $(y, s) \in B$ as above, pick s_0 the minimal s. If $s = s_0$, we are done since then the cone has a tip. We claim $s_0 > 0$ is impossible. For assume to get a contradiction that $s_0 > 0$. Pick $(z, t) \in B$ with $t < s_0$. The point z is in the finite subcover. Let B be the ball in that cover containing z , it comes with a point $(x, y) \not\in B$ and $(y, s) \in B$ with

   $$r + s + \epsilon \leq d(x, y)$$

   Then

   $$r + t + \epsilon \geq d(x, z) + d(y, z) \geq d(x, y) \geq r + s + \epsilon$$

   This is a contardiction sicnce $t < s_0 \leq s$.

2. Assuem C does not cover K, and let z be a point not covered. Then there does not exist $(y, s) \in B$ with

   $$d(z, y) \geq r$$

   Hence the cone is of the form $B(z)$

The following definition is a property between separability and compactness.

Definition 3 *A subset M' of a metric space M is called totally bounded, if for every $r > 0$ there exists a cover of M by balls of radius r.*

Lemma 5 *A totally bounded space is separable.*

Proof: For each r pick a finite cover C_r of the space M. Let C be any cover of M. Let x be a point in M. and $B_r(y)$ a ball in C containing x. then there is a ball $B_s(x)$ contained in $B_r(y)$. Pick a ball B^x in $C_{s/2}$ containing x, then this ball is contained in $B_r(y)$. The collection of all B^x is countable since contained in the countable union of finite sets C_r. Picking for each ball B^x a ball in C that contains B^x gievs a countable subcover of C. $\square$

Lemma 6 *A compact metric space is totally bounded.*

Proof: Apply the definiiton of compactness to the collection of balls of radius r. $\square$

Lemma 7 *Let (K, d) be a totally bounded complete metric space Then K is compact.*

Proof:

Denote by C_n a finite cover of M by balls of radius 2^{-n}. Pick a ball in $C_{2^{-0}}$ that is not covered by finitely many balls in C. Assume we have pickead a ball in $C_{2^{-n}}$ not covered by finitely balls in C, then cover this balls by finitely many balls in $C_{2^{-n-1}}$ and pick one such ball which is not covered by finitely many balls in C.

This gives a sequence of balls. Let x_n be the sequence of centers and note that $(x_n, 22^{-n})$ is a pre-cone. Let x be the tip of the corresponding cone.

Let $B_r(y)$ be a ball in C that contains x. Then this ball contains a ball $B_s(x)$. Then this ball contains the selected ball $C_{2^{-n}}$ for some n with $2^{-n} < s/2$. This is a contradiction to the choice of these selected balls.

$\square$

Totally boundedness relates to Cauchy sequences as follows:

Lemma 8 *Let M be a totally bounded metric space. Then every sequence has a subsequence which is Cauchy.*

Proof: Let $f : \mathbf{N} \to M$ be a sequence . define $f_0 = f$. Assume we have defined f_n for $n \geq 0$. Define a subsequecne f_{n+1} of f_n so that $f_{n+1}(m)$ coincides with $f\_n(m)$ for $m \leq n$, and so that $f_{n+1}(m) \in B_{2^{-n}}(x)$ for some ball of radius 2^{-n}. This is possible, since we can cover the space M by finitely many balls, so at least one of them needs to contain infinitely many values of the sequence f_n. Then define $g(n) = f_n(n)$. This is a subsequecne of f, and it is cauchy. $\square$

Lemma 9 *Let M be a metric space. Assume that every sequence $f : \mathbf{N} \to M$ has a subsequence that is Cauchy, then M is totally bounded.*

Proof: Fix r and pick a point $x_0 \in M$. Assume we have already picked x_n. If the collection of balls $B_r(x_k)$ with $0 \leq k \leq n$ covers M, we are done having constructed a finite cover of M. If not, pick x_{n+1} not covered by these balls. We claim this process cannot go on indefinitely. If it does, then we have a seuqence x_n which has no Cauchy subsequecne since any two elements in the seuqence have distance at least r. this is a contradiction. $\square$

Lemma 10 *Let (M, d) be a compact metric space and M' be a subspace. If M/M' is the union of balls, then M' is compact.*

Proof: Let C be a cover of M'. Since $M \setminus M'$ is the union of balls, the set C' of these balls forms a cover of $M \setminus M'$. Hence $C \cup C'$ covers M. Find a finite subcover. Remove all balls in C' from this finite subcover, then it still covers M' $\square$

Exercise 1 *Let (M, d) be a complete metric space and M' be a subspace. If M/M' is the union of balls, then M' is complete.*

Note that $\mathbf{R}^n$ with Euclidean metric is complete.

Lemma 11 *Bounded complete subsets of $\mathbf{R}^n$ with Euclidean metric are compact.*

Proof: We prove that every sequence in K has a subsequence which is Cauchy. let $(x_1, \ldots, x_n)$ denote the sequence. Since the sequence is bounded, each entry is bounded. Choose a subsequence so that x_1 is monmotone. Then choose a subsequence so that x_2 is monotone, and so on. Hence we may assume all sequences are monotone. But then all coordinate sequences are Cauchy, and hence the full sequence is Cauchy. Since K is complete, it has a limit.

□

Exercise 2 *The space $l^2(\mathbf{N})$ is separable and complete.*

Lemma 12 *The ball $B_1(0)$ in $l^2(\mathbf{N})$ is not compact.*

Proof: Let f_n be the element in $l^2(\mathbf{N})$ which satisfies $f_n(m) = 0$ if $n \neq m$ and $f_n(m) = 1$ if $n = m$. Then Any two elements have distance > 1, hence this sequence has no subsequence that is Cauchy.

Exercise 3 *Let $a : \mathbf{N} \to \mathbf{R}_0^+$ be a sequence with $\limsup_{n\to\infty} a_n = 0$. Let K be the set of all sequences f in $l^2(\mathbf{N})$ (with values in $\mathbf{R}$) such that*

$$\sum_{n=0}^{\infty} a(n) f(n)^2 < \infty < \infty$$

Prove that K is compact.

Exercise 4 *Prove the converse of the above. If $K \subset l^2(\mathbf{N})$ is compact, then there exists a sequence a_n as above such that $\sum_{n=0}^{\infty} a(n) f(n)^2 < \infty$ for all $f \in K$*

The following is a version of the fact that finite subsets in a totally ordered space have a maximum and a minimum.

Lemma 13 *Let (M, d) be a metric space and let K be a subspace. Let $x \in M$, then there exists y and z in K such that*

$$d(x, y) = \inf_{t \in K} d(x, t)$$

$$d(x, y) = \sup_{t \in K} d(x, t)$$

Proof: There is a sequence t_n of elements in K such that $d(x, t_n)$ is monotone increasing and

$$\sup_n d(x, t_n) = \sup_{t \in K} d(x, t) = S$$

Pick a subsequence that is Cauchy. Then this seqeunce has a limit $t \in K$. Then we have

$$d(x, t) \geq d(x, t_n) - d(t, t_n) \geq S - \epsilon - \epsilon$$

for arbitrarily small ϵ. Hence

$$d(x, t) = S$$

The proof of existence of a minimum is similar. □

Definition 4 *Let (M, d) and (M', d') be two metric spaces. The Cartesian product of the two metric spaces is the set of all ordered pairs (x, x') with $x \in M$ and $x' \in M'$ and distance*

$$d((x, x'), (y, y')) = \max\{d(x, y), d(x, y')\}$$

Exercise 5 *Prove that this is indeed a metric space.*

Exercise 6 *The Cartesian product of two separable metric spaces is separable.*

Lemma 14 *The Cartesian product of two compact metric spaces is compact.*

Proof: It suffices to find finite subcovers of covers by balls in this metric. Suppose we are given such a cover. For each $x \in M$ this covers in particular the set $\{x\} \times M'$. Find a finite subcover. Pick the minimal radius r of this finite subcover. Then this finite subcover covers $B_r(x) \times M'$. Now these $B_r(x)$ cover all of M. Find a finite subcover. This produces a finite colelction of ifnite sets covering all of $M \times M'$ □

Lemma 15 *Assume (K, d) is a compact metric space and (M, d') is a metric space. Assume $f : K \to M$ is continuous. Then the range of f is compact.*

Proof: Let C be an open cover of $f(K)$. We produce a cover of K. Let $x \in K$. Let B'_x be a ball in C that contains $f(x)$. By continuity of f, there is a ball B_x in K such that $f(B_x) \subset B'_x$. The collection of these balls B_x covers K. Pick a finite subcover. Consider all B'_x corresponding to the chosen B_x. This is a finite collection. We claim it covers $f(K)$. Let $y \in f(K)$. The there is z such that $f(z) = y$. Let B_x be the chosen ball containing z. Then B'_x contains y. □

Lemma 16 *Let (K, d) be a compact space and (M, d') be a metric space. Let $f : K \to M$ be continuous. then f is uniformly continuous.*

Fix $s > 0$. define for each $x \in K$ $r(x)$ to be the supremum of all numbers $r \leq 1$ such $d(x, y) \leq r$ implies $d'(f(x), f(y)) \leq \delta$. We claim that r is continuous, and first see how this will finish the proof. If continuous, then it attains its minimum on the compact set K. The minimum cannot be 0 by continuity at that point. Hecne the minimu is larger than zero, which proves uniform continuity since s was arbitrary.

Now we turn to the proof of continuity. Fix x and note that in the vicinity oif x we will use.

□

Arzela Ascoli

Lemma 17 *Let M and M' be compact spaces. Assume $F \subset C(M, M')$. Assume that there is a modulus of continuity g such that all $f \in F$ satisfy*

$$g(d'(f(x), f(y)) \leq d(x, y)$$

Then F is totally bounded.

Proof:

Pick $r > 0$. Pick $s = g(r/4)$. Cover K by finitely many balls $B_s(x_i)$, $i = 1, \ldots, n$. For each i cover the compact set $\{f(x_i), f \in F\}$ by finitely many balls $B_{r/4}(y_{i,j})$, $j = 1, \ldots m$.

For each of the finitely many functions

$$h : \{1, \ldots, n\} \to \{1, \ldots, m\}$$

for which there exists a function $f \in F$ with

$$f(x_i) \in B_{r/4}(y_{i,h(i)})$$

Pick one of these functions and call it f_h. This is a finite collection in F. We claim the corresponding balls of radius r about these functions in $C(M, M')$ cover F.

To see this, let f be any function in F. Pick a function h such that

$$f(x_i) \in B_{r/4}(y_{i,h(i)})$$

which exists by construction. We aim to show for all $x \in M$ that

$$d(f(x), f_h(x)) \leq r$$

Given such x, there is x_i with $x \in B_s(x_i)$. Then

$$d(f(x), f'(x)) \leq d(f(x), f(x_i)) + d(f(x_i), y_{i,h(i)}) + d(y_{i,h(i)}, f_h(x_i)) + d(f_h(x_i), f_h(x))$$

But each of these numebers are less than $r/4$ by construction. This completes the proof.

□

The Riemann integral for continuous functions

So far we have defined the integral

$$\int_0^x f(t)\, dt$$

for monotone increasing functions. It satisfies

$$\int_0^x (f(t) + g(t))\, dt = \int_0^x f(t)\, dt + \int_0^x g(t)\, dt$$

and

$$\int_0^x f(t)\,dt \leq \int_0^x g(t)\,dt$$

whenever $f \leq g$.

We can extend this to the space of fucntions of bounded variation. This is the set of pairs (f, g) of monotone increasing functions with equivalence relation $(f, g) \sim (f', g')$ if $f + g' = f' + g$.

Recall that these functions take value in the rela numbers, which is the set of pairs (x, y) with the equivalence relation $(x, y) \sim (x', y')$ if $x + y' = x' + y$

The Riemann integral is then defined as

$$\int_0^x (f(t), g(t))\,dt = (\int_0^x f(t)\,dt, \int_0^x g(t)\,dt)$$

It does not depend on the choice of representative of the equivalence class, since for any two representatives $(f, g), (f', g')$ with $f + g' = f' + g$ we have

$$\int_0^x f(t)\,dt + \int_0^x g'(t)\,dt = \int_0^x f(t) + g'(t)\,dt$$

$$\int_0^x f'(t) + g(t)\,dt = \int_0^x f'(t)\,dt + \int_0^x g(t)\,dt$$

and hence

$$(\int_0^x f(t)\,dt, \int_0^x g(t)\,dt) \sim (\int_0^x f'(t)\,dt, \int_0^x g'(t)\,dt)$$

We refer to introduction of rela numbers to elsewhere in the script and use their properties.

We like to extend this integral on $BV([0, x])$ to the space $C([0, x], \mathbf{R})$. More precisely, neither of the spaces contains the other, even after natural identification of $(f, g) \in BV([0, x])$ with the function $f - g \in C([0, x])$. However, the space of Lipschitz functions is in the inetrsection of both. Clearly Lipschitz functions are continuous. Moreover, if f is Lipschitz with constant L, then we can write it as difference of twpo monotone functions as follows:

$$f(t) = (f(t) + Lt) - Lt$$

The Lipschitz condition implies the first function is monotone.

Recall that the Lipschitz functions are dense in $C[0, x])$.

The point is that for Lipschitz functions we have the continuity property (actual even a Lipschitz cotinuity)

$$|\int_0^x f(t)\,dt - \int_0^x g(t)\,dt| \leq x\|f - g\|_\infty$$

This can be seen as follows. Let L be the maximum of the Lipschitz constants for f and g. Then we write

$$f(t) = (f(t) + L(t)) - L(t)$$

$$g(t) = (g(t) + L(t)) - L(t)$$

Representing each as monotone function. Then the LHS can be seen to be

$$|\int_0^x f(t) + Lt\,dt - \int_0^x g(t) + Lt\,dt|$$

Denote $\|f - g\|_\infty$ by d, then

$$\int_0^x f(t) + Lt\,dt \leq \int_0^x g(t) + Lt + d\,dt = \int_0^x g(t) + Lt\,dt + xd$$

$$\int_0^x g(t) + Lt\,dt \leq \int_0^x f(t) + Lt + d\,dt = \int_0^x f(t) + Lt\,dt + xd$$

This proves the desired Lipschitz continuity.

Since Lipschitz functions are dense in $C([0,x])$, there is a unique extension to $C([0,x])$ if this Lipschitz mapping. This defines

$$\int_0^x f(t)\,dt$$

for functions f in $C([0,1])$

Similarly one can see that the n-th Lower and upper Riemann sums can be extended to $C([0,1])$. Moreover, all the n-th Riemann sums for a fixed function are bounded by $\|f\|_\infty x$ and Lipschitz with constant x. This makes them an equicontinuous family.

Since for every Lipschitz function f the Riemann sums converge to the integral, so do the Riemann sums for arbitrary functions in $C([0,1])$ (use a theorem about uniqueness of the limit.)

Let $I : C([0,1]) \to C([0,1])$ be defined by

$$If(x) = \int_0^x f(t)\,dt$$

First of all, we note that indeed If defines a continuous function, indeed, it is even Lipschitz. But if $x < y$ we have

$$If(y) - If(x) = \int_x^y f(t)\,dt \leq |y - x|\|f\|_\infty$$

Hence If is even a Lipschitz function.

We like to understand properties of the range of the map I.

Definition 5 *We call a function $g : [0,1] \to \mathbf{R}$ to be continuosuly differentiable, if for each $x \in [0,1)$ the limit*

$$g(x) = \lim_{n\to\infty} \frac{g(x + 2^{-n}) - g(x)}{2^{-n}}$$

exists and g is a uniformly continuous function on $[0,1)$.

Lemma 18 *The range of I is contained in the space of continuously differentiable functions. I.e., If f is continuous and $g = If$, then for each $x \in [0,1)$ we have*

$$\lim_{n\to\infty} \frac{If(x+2^{-n}) - If(x)}{2^{-n}} = f(x)$$

Proof: Note if n is large enough then

$$2^{-n}(f(x)-\epsilon) \leq If(x+2^{-n}) - If(x) \leq 2^{-n}(f(x)+\epsilon)$$

This proves existence of the limit. Since f is uniformly contiunuous, this proves the lemma. $\square$

Lemma 19 *The map I is injective.*

Proof: Suppose $If = Ig$. Then $I(f-g) = 0$. Hence it suffices to show $Ih = 0$ implies $h = 0$. Assume to get a contradictiopn $h \neq 0$, then there exists $y \in (0,1)$ with $h(y) \neq 0$. Assume wlog $h(y) > \epsilon > 0$. Then there is a ball of radius δ about y with $h(x) > \epsilon$ in this ball. But then

$$Ifh(y-\delta) + 2\epsilon\delta \leq Ih(y+\delta)$$

And thus $IH \neq 0$. thsi is a contradiction. $\square$

Lemma 20 *The map I is surjective.*

Proof: Let g be some continuously differentiable function, and let f denote its continuosu derivative (extended to $[0,1]$). It suffices to show $g = If$, i.e., for every

$$\int_0^x f(t)\,dt = g(x)$$

...
$\square$