MIDTERM 2 (131A)

Name:

1	2	3	4	5	total

Problem 1.

1) Define what it means for a function $f : \mathbf{R} \to \mathbf{R}$ to be continuous at a point $x_0 \in \mathbf{R}$.

2) Suppose $f : \mathbf{R} \to \mathbf{R}$ is continuous at $x_0 \in \mathbf{R}$ and $f(x_0) > 0$. Prove that there exists an $\epsilon > 0$ such that for all $x \in \mathbf{R}$ with $|x - x_0| < \epsilon$ we have $f(x) \neq 0$.

Solution:

1) The function f is continuous at x_0 if for every $\epsilon > 0$ there exists a $\delta > 0$ such that for all x with $|x - x_0| \leq \delta$ we have $|f(x) - f(x_0)| \leq \epsilon$

2) Fix x_0 as in the problem. Set $\epsilon_1 = f(x_0)/2$ which is a positive number. The definition of continity gives a $\delta > 0$ such that for all x with $|x - x_0| \leq \delta$ we have $|f(x) - f(x_0)| < f(x_0)/2$. In particular $f(x)$ cannot be zero for such x because $|0 - f(x_0)| = f(x_0) > f(x_0)/2$. Choosing ϵ to be this δ finishes the proof.

Problem 2.

1) Define what it means for a function to be uniformly continuous.
2) Give a domain $D \subset \mathbf{R}$ and a function on D that is continuous but not uniformly continuous on D. Prove both assertions (continuity and non-uniform continuity).

Solution:

1)
A function $f : D \to \mathbf{R}$ is uniformly continuous if for every $\epsilon > 0$ there exists a $\delta > 0$ such that for all $x, y \in D$ with $|x - y| \leq \delta$ we have $|f(x) - f(y)| \leq \epsilon$.

2)
Let $D = \mathbf{R} \setminus \{0\}$. Define $f : D \to \mathbf{R}$ by $f(x) = -1$ if $x < 0$ and $f(x) = 1$ if $x > 0$.

a) f is continuous at every point in the domain: Given $x \in D$ and $\epsilon > 0$ choose $\delta = |x|/2 > 0$, then for every $y \in D$ with $|x - y| \leq |x|/2$ we have that x and y are on the same side of the origin because if $x > 0$ we have

$$y = x - (x - y) \geq x - |x - y| \geq x - |x|/2 = x/2 > 0$$

and if $x < 0$ we have

$$y = x + y - x \leq x + |y - x| \leq x + |x|/2 = x/2 < 0$$

Hence $|f(x) - f(y)| = 0 \leq \epsilon$

b) The negation of uniform continuity is: There exists $\epsilon > 0$ such that for all $\delta > 0$ there exists $x, y \in D$ such that $|x - y| \leq \delta$ and $|f(x) - f(y)| > \epsilon$.

Choose $\epsilon = 1$. For any $\delta > 0$ choose $x = -\delta/4$ and $y = \delta/4$. Then $|x - y| = \delta/2 < \delta$ and $|f(x) - f(y)| = 2 > 1 = \epsilon$.

Problem 3.

Define 0.9999... to be the series

$$\sum_{k=1}^{\infty} \frac{9}{10^k}$$

1) Prove that 0.9999... converges.
2) Prove that 0.9999... is equal to one.

Solution (both 1 and 2, this solution is very detailed, some of the steps used in this argument would have been OK if simply cited): Consider the partial sums

$$\begin{aligned} s_n &= \sum_{k=1}^{n} \frac{9}{10^k} \\ &= 9\left(\sum_{k=1}^{n} \frac{1}{10^k}\right) \\ &= \frac{9}{10}\left(\frac{1-(1/10)^n}{1-(1/10)}\right) \\ &= 1-(1/10)^n \end{aligned}$$

Here have used that if we set

$$b = \left(\sum_{k=1}^{n} \frac{1}{10^k}\right)$$

we have

$$b(1-1/10) = 1/10-(1/10)^{n+1}$$

which can be solved for b.

We know that

$$\lim_{n\to\infty} (1/10)^n$$

exists, because $(1/10)^n$ is a monotone decreasing sequence of positive numbers. Hence

$$\lim_{n\to\infty} s_n$$

exists.

Morover, if we set

$$a = \lim_{n\to\infty} (1/10)^n =$$

we have by shifting the sequence

$$a = \lim_{n\to\infty} (1/10)^{n+1} = \frac{a}{10}a$$

and the only solution to that is $a = 0$.

Hence

$$\lim_{n\to\infty} s_n = 1$$

Problem 4.

Prove the comparison test: If a_n and b_n are two sequences of real numbers with $|a_n| \leq b_n$ for all n, and if

$$\sum_{n=1}^{\infty} b_n$$

converges, then

$$\sum_{n=1}^{\infty} a_n$$

converges as well.

Define the partial sums

$$s_n = \sum_{n=1}^{n} a_k$$

and

$$t_n = \sum_{n=1}^{n} b_k$$

The sequence $t - n$ converges, and hence is Cauchy. Hence for each $\epsilon > 0$ there exists N such that for all $n > m > N$ we have $|t_n - t_m| \leq \epsilon$

For this same ϵ, N, n, m we have with the triangle inequality

$$|s_n - s_m| = |\sum_{k=m+1}^{n} a_k| \leq \sum_{k=m+1}^{n} |a_k| \leq \sum_{k=m+1}^{n} b_k = |t_n - t_m| \leq \epsilon$$

Hence s_n is also Cauchy. Since every cauchy sequence converges, s_n converges.

Problem 5.

Let x_n be a sequence of real numbers that is bounded above.

1) Define $\limsup(x_n)$.

2) Prove that there exists a subsequence of the sequence x_n which converges to $\limsup(x_n)$.

Solution:

1) Define $y_n = \sup\{x_m : m \geq n\}$, then

$$\limsup(x_n) = \inf\{y_n\}$$

2) We need to find a strictly monotone increasing map $\mathbf{N} \to \mathbf{N}$, $m \mapsto n_m$ so that the sequence $t_m = x_{n_m}$ converges to $\limsup(x_n)$.

We define $m_1 = 1$. Assume we have already defined $m_1, \ldots, m_n$, we plan to define m_{n+1}.

Given m, we can find an $k > n_m$ such that

$$\limsup(x_n) \leq y_k \leq \limsup(x_n) + 1/m$$

Namely, the first inequality holds for all k by definition of inf and the second inequality holds for some $k \geq 1$ because $\limsup(x_n) + 1/m$ is not a lower bound for the set $\{y_n\}$ and then holds also for some possibly other $k > n_m$ because the sequence y_n is decreasing.

Given that k, we can find an $n_{m+1} \geq k$ such

$$y_k - 1/m < x_{n_{m+1}} < y_k$$

where we use the definition of y_k is the supremum of the set of all x_n with $n \geq k$. Hence we have

$$\limsup(x_n) - 1/m \leq y_k - 1/m < x_{n_{m+1}} < y_k \leq \limsup(x_n) + 1/m$$

or equivalently

$$|x_{n_{m+1}} - \limsup(x_n)| \leq 1/m$$

Hence the subsequence $t_m = x_{n_m}$ converges to $\limsup(x_n)$.