# Delta and Heaviside functions

δ-functions are limiting cases of differentiable/continuous functions.

for example,
$$f(x) = \int_{-\infty}^{+\infty} dx' \; g(x-x') \; f(x')$$

$$g(x-x') \equiv \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\lambda \; e^{i\lambda(x'-x)} \longrightarrow \delta(x-x')$$

↳ picks off the value at X.

δ-function is infinitely narrow and infinitely high such that $\int \delta(x)\, dx = 1$

Derivative of δ-function?

$$\int_{-\infty}^{+\infty} f(x') \frac{d}{dx'} \delta(x'-x) \; dx' = \int_{-\infty}^{+\infty} f(x') \frac{d\,\delta(x'-x)}{dx'} \; dx'$$

Integrating by parts,
$$\delta(x'-x) f(x') \Big|_{x'=-\infty}^{x'=\infty} - \int_{-\infty}^{+\infty} \delta(x'-x) \frac{\partial}{\partial x'} f(x') \, dx'$$

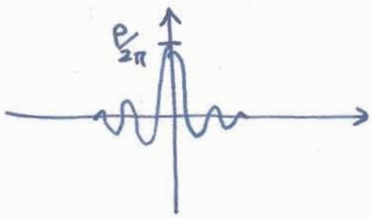
$$= -f'(x)$$

# Representation of δ-function

$$\frac{1}{2\pi}\int_{-P/2}^{P/2} d\lambda \,\cos\lambda(x'-x) = \frac{1}{2\pi}\,\frac{\sin\lambda(x'-x)}{(x'-x)}\Bigg|_{\lambda=-P/2}^{\lambda=P/2}$$

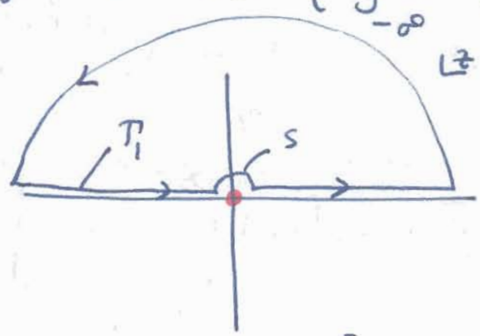
$$= \frac{1}{\pi}\,\frac{\sin\frac{P}{2}(x'-x)}{(x'-x)} \xrightarrow[P\to\infty]{}$$

if $x' \neq x$,




check for area: $\displaystyle\int_{-\infty}^{+\infty} \frac{1}{\pi}\,\frac{\sin\frac{P}{2}y}{y}\,dy = \frac{2}{\pi}\int_0^{\infty} \frac{\sin\frac{Py}{2}}{\frac{Py}{2}}\,d\left(\frac{Py}{2}\right)$

note that this integral is $\mathrm{Im}\left\{\int_{-\infty}^{+\infty}\frac{e^{iz}}{z}\,dz\right\}$




$$\oint_c \frac{e^{iz}}{z}\,dz = 0, \quad \therefore \underbrace{\oint_{\Gamma_1}\frac{e^{iz}}{z}\,dz}_{\int_{-\infty}^{-\varepsilon}\frac{e^{iz}}{z}dz + \int_{\varepsilon}^{\infty}\frac{e^{iz}}{z}dz} = -\oint_s \frac{e^{iz}}{z}\,dz$$

$$= -\int_{\pi}^{0} d\theta\, i\cancel{\varepsilon e^{i\theta}}\,\frac{e^{i\varepsilon e^{i\theta}}}{\cancel{\varepsilon e^{i\theta}}} = i\pi$$

Im part $=\pi$, $\displaystyle\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\sin\frac{P}{2}y}{y}\,dy = 1$ ✓

Another representation:

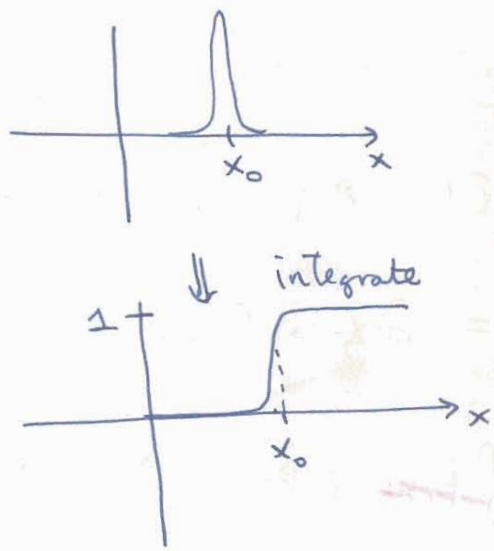
$$\lim_{p\to\infty} \frac{p}{\pi(1+p^2x^2)} \longrightarrow \delta(x)$$

for $x \neq 0$, $\sim \dfrac{1}{\pi p x^2} \underset{p\to\infty}{\longrightarrow} 0$

for $x = 0$, $\sim \dfrac{p}{\pi} \underset{p\to\infty}{\longrightarrow} \infty$

yet another: $\delta_p(x) = \dfrac{p}{2\cosh^2 px} \begin{cases} \to 0 & \text{if } x \neq 0 \\ \to \infty & \text{if } x = 0 \end{cases}$

What is the integral of a $\delta(x)$?

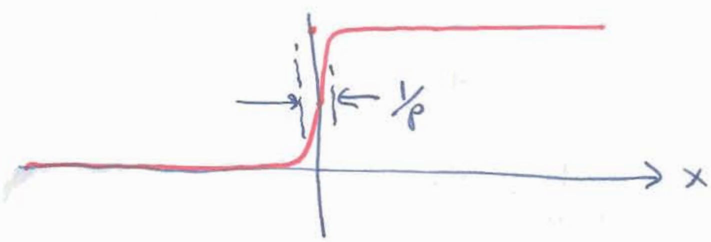



$$\int_{-\infty}^{x} \frac{p}{2\cosh^2 px'}\,dx = \frac{1}{2}\int_{-\infty}^{px} d\xi\, \frac{1}{\cosh^2\xi}$$

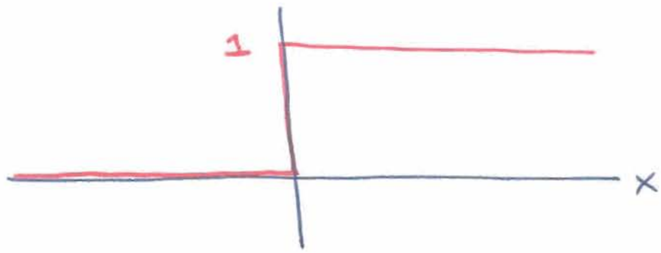
$$= 2\int_{-\infty}^{px} \frac{d\xi}{(e^{\xi}+e^{-\xi})^2}, \quad e^{\xi} \equiv y, \quad d\xi = \frac{dy}{y}$$

$$= \int_0^{e^{px}} \frac{d(y^2)}{(y^2+1)^2} \Longrightarrow \int_0^{e^{2px}} \frac{dy}{(y+1)^2}$$

$$= \int_1^{e^{2px}+1} \frac{d\alpha}{\alpha^2} = -\frac{1}{e^{2px}+1} + 1 = \frac{e^{px}}{e^{px}+e^{-px}} = \frac{1}{2}\left[1+\tanh px\right]$$

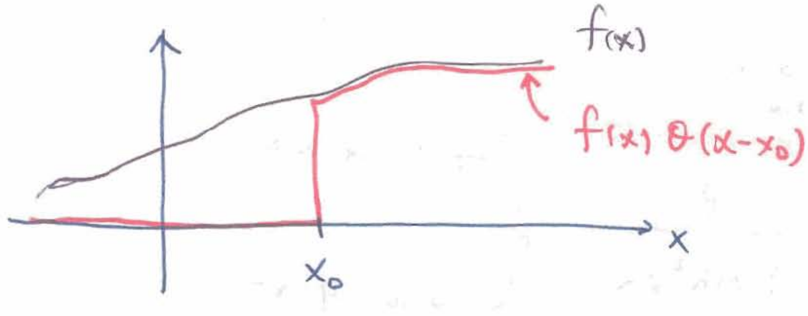


As $p \to \infty$, this approaches a "step" function:




$\theta(x) \equiv$ Heaviside function

$$\theta(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$$




Convention: $\theta(0) = \frac{1}{2}$

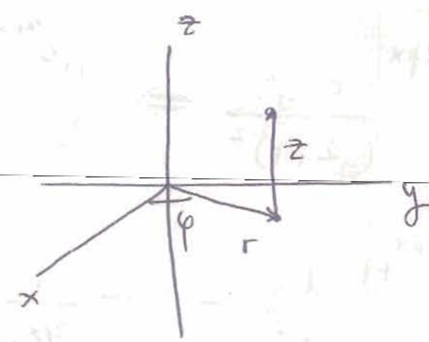
Multidimensional $\delta$-functions

$\delta(\vec{r} - \vec{r}_0)$ what are units of $\delta$-function?

$$= \delta(x - x_0)\,\delta(y - y_0)\,\delta(z - z_0)$$

$$\int f(\vec{r}\,')\,\delta(\vec{r} - \vec{r}\,')\,d^3\vec{r}\,' = f(\vec{r})$$

in Cylindrical Coordinates




$$d\vec{r} = \hat{r}\,dr + \hat{\varphi}\,r\,d\varphi + \hat{k}\,dz$$

$$\nabla = \hat{r}\,\partial_r + \frac{1}{r}\hat{\varphi}\,\partial_\varphi + \hat{k}\,\partial_z$$

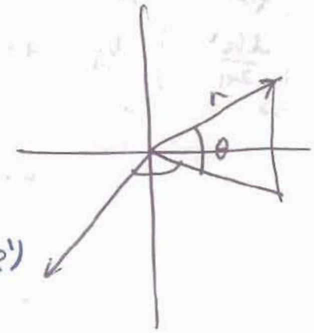
$$d^2\vec{r} = r\,d\varphi\,dz$$

$$\int d^2\vec{r}\,'\,\delta(\vec{r} - \vec{r}\,') = \int \delta(\vec{r} - \vec{r}\,')\,r'\,d\varphi'\,dz' \equiv 1$$

$$\therefore\ \delta(\vec{r} - \vec{r}\,') = \frac{1}{|\vec{r}|}\,\delta(r - r')\,\delta(\varphi - \varphi')\,\delta(z - z')$$

## Spherical coordinates

$$d^3\vec{r} = r^2 dr\, d\Omega = r^2 dr \sin\theta\, d\theta\, d\varphi$$




$$\therefore\ \delta(\vec{r}-\vec{r}\,') = \frac{1}{|\vec{r}|^2}\frac{1}{\sin\theta}\,\delta(r-r')\,\delta(\theta-\theta')\,\delta(\varphi-\varphi')$$

## Properties of Fourier Transform

⊛ Linearity
$$\int_{-\infty}^{+\infty}[a f(x) + b g(x)]\,e^{-ikx}\,dx = a F(k) + b G(k)$$

⊛ Scaling
$$\int_{-\infty}^{+\infty} f(ax)\,e^{-ikx}\,dx = \int_{-\infty}^{+\infty} f(y)\,e^{-i\frac{ky}{a}}\,\frac{dy}{a} = \frac{1}{a}F\left(\frac{k}{a}\right)$$

⊛ Multidimensional
$$\prod_{i=1}^{N}\int_{-\infty}^{+\infty} f(x_1,x_2,x_3\ldots x_N)\,e^{-i(k_1x_1+k_2x_2+\ldots k_Nx_N)}\,dx_i$$

⊛ Convolution
$$f * g\,(x) \equiv \int_{-\infty}^{+\infty} g(y)\,f(x-y)\,dy = ?$$

$$g(y) = \int_{-\infty}^{+\infty} G(k)\,e^{iky}\,\frac{dk}{2\pi}$$

$$= \int_{-\infty}^{+\infty} F(k')\,e^{ik'(x-y)}\,\frac{dk'}{2\pi}$$

$$[f*g](x) = \int_{-\infty}^{+\infty} dy\left(\int_{-\infty}^{+\infty} G(k)\,e^{iky}\,\frac{dk}{2\pi}\right)\left(\int_{-\infty}^{+\infty} F(k')\,e^{ik'(x-y)}\,\frac{dk'}{2\pi}\right) =$$

$$= \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \int_{-\infty}^{+\infty} \frac{dk'}{2\pi} \underbrace{\int_{-\infty}^{+\infty} dy \; G(k) F(k') e^{iy(k-k')}}_{} e^{ik'x}$$

$$\int_{-\infty}^{+\infty} dy \; e^{iy(k-k')} \Rightarrow 2\pi\delta(k-k')$$

do the $dk'$ integral,

$$[f*g](x) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} G(k) F(k) e^{ikx} \equiv \text{Fourier transform of } G(k)F(k).$$

Inverting, $\int_{-\infty}^{+\infty} [f*g](x) \, e^{-ikx} dx \equiv G(k) F(k)$

Fourier transform of convolution = product of the Fourier transforms of each function.

What is $\int_{-\infty}^{+\infty} e^{+ikx} F(k) G(k) H(k) \frac{dk}{2\pi}$ ?

Parseval's relation

$$\int_{-\infty}^{+\infty} g(y) f(-y) \, dy = \int_{-\infty}^{+\infty} G(k) F(k) \frac{dk}{2\pi}$$

⊛ Real functions

if $f(x) \in \mathbb{R}$ $\quad \int_{-\infty}^{+\infty} F(k) e^{ikx} \frac{dk}{2\pi}$ is real,

$k \to -k$, $\quad \int_{-\infty}^{+\infty} F(-k) e^{-ikx} \frac{dk}{2\pi} \equiv \int_{-\infty}^{+\infty} F^*(k) e^{-ikx} \frac{dk}{2\pi}$

$\boxed{\therefore \text{ for real } f(x) \quad F^*(k) = F(-k)}$

## Derivatives & Fourier transforms



$$\int_{-\infty}^{+\infty}\left(\frac{\partial f}{\partial x}\right)e^{-ikx}\,dx \xrightarrow[\text{integration by parts}]{} f(x)\,e^{-ikx}\Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} -ik\, f(x)\,e^{-ikx}\,dx$$

$$= 0 + ik\int_{-\infty}^{+\infty} f(x)\,e^{-ikx}\,dx$$

$$\equiv ik\,F(k)$$

In general $\displaystyle\int_{-\infty}^{+\infty}\left(\frac{\partial f}{\partial x_i}\right)e^{-i\underbrace{(k_1x_1+k_2x_2+\ldots)}_{\vec{k}\cdot\vec{x}}}\,dx = ik_i\,F(k_1,k_2,\ldots k_N)$

<u>higher derivatives</u>: $\displaystyle\int_{-\infty}^{+\infty}\left[\frac{\partial^n f(x)}{\partial x^n}\right]e^{-ikx}\,dx = (ik)^n F(k)$

<u>vector derivatives</u>: gradient $\vec{\nabla}$: 

$$\int_{-\infty}^{+\infty}(\vec{\nabla} f)\,e^{-i\vec{k}\cdot\vec{x}}\,dx_1\,dx_2\ldots dx_N$$

$$= i(\hat{x}_1k_1+\hat{x}_2k_2+\ldots\hat{x}_Nk_N)\,F(\vec{k}) = i\vec{k}\,F(\vec{k})$$

laplacian $\nabla^2$:

$$\int_{-\infty}^{+\infty}(\nabla^2 f)\,e^{-i\vec{k}\cdot\vec{x}}\,d\vec{x} = -(\vec{k}\cdot\vec{k})F(\vec{k})$$

FT turns derivatives to multiplication w/ $ik$ $\Rightarrow$ solve differential eqns.

## Diffusion-convection equation

$$\frac{\partial c(\vec{x},t)}{\partial t} = -\vec{V}\cdot\vec{\nabla}c + D\nabla^2 c(x,t)$$

Fourier transform eqn:

$$\int_{-\infty}^{+\infty}\left[\frac{\partial c(\vec{x},t)}{\partial t} = -\vec{V}\cdot\vec{\nabla}c + D\nabla^2 c(\vec{x},t)\right] e^{-i\vec{k}\cdot\vec{x}}\, d^d\vec{x}$$

$$\equiv \frac{\partial \tilde{c}(\vec{k},t)}{\partial t} = -\vec{V}\cdot(i\vec{k})\,\tilde{c}(\vec{k},t) - Dk^2\tilde{c}(\vec{k},t)$$

$$= -\left[i\vec{V}\cdot\vec{k} + Dk^2\right]\tilde{c}(\vec{k},t)$$

Now solve the ODE in time:

$$\tilde{c}(\vec{k},t) = \tilde{c}(\vec{k},0)\, e^{-[i\vec{V}\cdot\vec{k} + Dk^2]t}$$

what is $\tilde{c}(\vec{k},0)$ ?

$$\tilde{c}(\vec{k},0) = \int c(\vec{x},0)\, e^{-i\vec{k}\cdot\vec{x}}\, d\vec{x}$$

if $C(\vec{x},0)$ starts as a point source,

$$C(\vec{x},0) = C_0\,\delta(\vec{x}), \quad \text{and} \quad \tilde{C}(\vec{k},0) = C_0$$

$$\therefore \tilde{c}(\vec{k},t) = C_0\, e^{-[i\vec{V}\cdot\vec{k} + Dk^2]t}$$

find $C(\vec{x},t)$ by inverting,

$$C(\vec{x},t) = \int C_0\, e^{-[i\vec{v}\cdot\vec{k} + Dk^2]t} \frac{d^dk}{(2\pi)^d}\, e^{i\vec{k}\cdot\vec{x}}$$

for 1D, $$C(x,t) = C_0 \int_{-\infty}^{+\infty} \frac{dk}{2\pi}\, e^{-iVkt - Dk^2 t}\, e^{ikx}$$

complete the square: $(k+a)^2 = k^2 + 2ak + a^2 =$

$$k^2 - i\frac{(x-vt)}{Dt}k + a^2$$

$$a = -\frac{i}{2}\frac{(x-vt)}{Dt}$$

$$C(x,t) = e^{-Dt\, a^2}\, C_0 \int_{-\infty}^{+\infty} \frac{dk'}{2\pi}\, e^{-Dt\, k'^2}$$

$$= C_0\, e^{-\frac{Dt}{4}\frac{(x-vt)^2}{(Dt)^2}} \cdot \frac{1}{\sqrt{4\pi Dt}}$$

$$= \frac{C_0}{\sqrt{4\pi Dt}}\, e^{-\frac{(x-vt)^2}{4Dt}}$$ $\Longrightarrow$ compare w/ result from probability diffusion.

for 3D $$C(\vec{x},t) = C_0 \int \frac{d^3k}{(2\pi)^3}\, e^{-i\vec{V}\cdot\vec{k}t - Dk^2 t}\, e^{i\vec{k}\cdot\vec{x}}$$

$$= C_0 \int \frac{k^2\, dk}{(2\pi)^3} \sin\theta\, d\theta\, d\varphi\; e^{+i\vec{k}\cdot\underbrace{(\vec{x}-\vec{v}t)}_{\text{reference axis } \hat{z}} - Dk^2 t}$$

$$\int_{-1}^{+1} e^{ik|\vec{x}-\vec{v}t|\cos\theta}\, d(\cos\theta) = \frac{2\sin(k|\vec{x}-\vec{v}t|)}{k|\vec{x}-\vec{v}t|}$$

$\int d\varphi \to 2\pi$

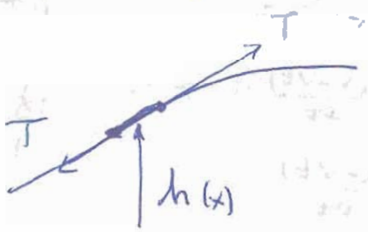
$$\int_0^{+\infty} \frac{k^2\, dk}{(2\pi)^2}\, e^{-Dt\,k^2}\, \frac{\sin k|\vec{x}-\vec{v}t|}{k|\vec{x}-\vec{v}t|} = \frac{2C_0\sqrt{\pi}}{(2\pi)^2\, 4(Dt)^{3/2}}\, e^{-|\vec{x}-\vec{v}t|^2/4Dt} = \boxed{\frac{1}{(4\pi Dt)^{3/2}}\, e^{-|\vec{x}-\vec{v}t|^2/4Dt}}$$

# Wave Equation

$$\ddot{\phi} = c^2 \frac{\partial^2 \phi}{\partial x^2}$$

String under tension

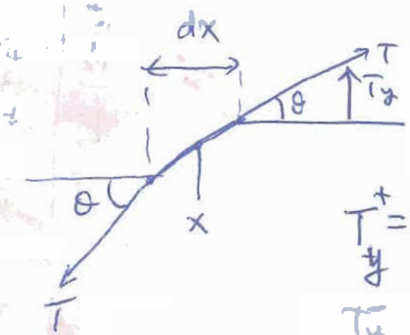

$\rho$ = mass/unit length




Local force balance:

$dx\, \rho\, \ddot{h}(x,t)$ = acceleration × mass

= sum of normal forces




$$T_y^+ = T \sin \theta\left(x + \frac{dx}{2}\right) \simeq T\, \theta\left(x + \frac{dx}{2}\right)$$

$$T_y^- = T \sin \theta\left(x - \frac{dx}{2}\right) \simeq T\, \theta\left(x - \frac{dx}{2}\right)$$

$+T_y^+ - T_y^-$ = sum of normal forces $= T\left(+\theta\left(x+\frac{dx}{2}\right) - \theta\left(x - \frac{dx}{2}\right)\right)$

$$\simeq +T\, dx\, \theta'(x) = +T \frac{d^2 h}{dx^2}$$

$$\boxed{\begin{gathered} \therefore\ \rho \ddot{h}(x,t) = +T \frac{d^2 h}{dx^2} \\ \ddot{h}(x,t) = \left(\frac{T}{\rho}\right) \frac{d^2 h}{dx^2} \quad \text{Wave equation} \end{gathered}}$$

$$\frac{T}{\rho} \equiv c^2 = \frac{\text{Force}}{\text{mass/length}} = \frac{\text{mass} \cdot \text{length}}{(\text{time})^2}\, \frac{\text{length}}{\text{mass}} = \frac{L^2}{T^2} = c^2$$

c = wave speed.