# Fourier Transforms

useful for solving some ODE's and PDE's

⊛ Taylor expansion : $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots$

⊛ Fourier expansion : expand $f(x)$ in powers of $e^{2\pi i s x} = \cos(2\pi s x) + i \sin(2\pi s x)$

for real $x$, $\boxed{f(x) = \int_{-\infty}^{+\infty} ds\ F(s)\, e^{2\pi i s x}\, ds}$

$F(s)$ is the Fourier transform of $f(x)$

⊛ If $x$ is time, then $s$ is frequency $\nu$;

$$f(t) = \int_{-\infty}^{+\infty} d\nu\ e^{2\pi i \nu t}\, F(\nu) \quad \text{or} \quad \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi}\, e^{i\omega t}\, F(\omega) \qquad (\omega = 2\pi\nu)$$

one can invert; $\boxed{F(s) = \int_{-\infty}^{+\infty} f(x)\, e^{-2\pi i s x}\, dx}$

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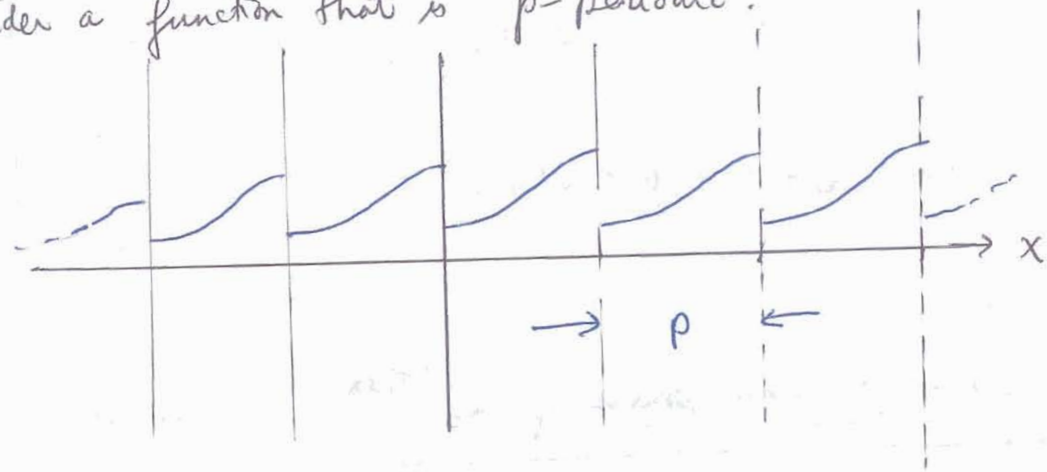
Both $f(x)$ and $F(s)$ have to decay as $s \to \pm\infty$, and $x \to \pm\infty$

<u>$F(s)$ and $f(x)$</u> are duals of each other

Consider a function that is $p$-periodic :




$$f(x+p) = f(x) = \int_{-\infty}^{+\infty} F(s)\, e^{2\pi i s (x+p)}\, ds$$

$$= \int_{-\infty}^{+\infty} F(s)\, e^{2\pi i s x} \underbrace{e^{2\pi i s p}}_{=1}\, ds \quad \therefore \; s = \frac{n}{p},\; n = 0, \pm 1, \pm 2, \ldots$$

$$f(x) = \int_{-\infty}^{+\infty} F(s)\, e^{2\pi i s x}\, ds = \sum_{n=-\infty}^{+\infty} F(n)\, e^{2\pi i \frac{n}{p} x} \quad \boxed{\text{Fourier Series}}$$

Another representation

$$e^{2\pi i \frac{n}{p} x} = \cos\left(\frac{2\pi n}{p} x\right) + i \sin\left(\frac{2\pi n}{p} x\right)$$

$$f(x) = F(0) + \sum_{n=1}^{\infty} F(n) \cos\left(\frac{2\pi n}{p} x\right) + \sum_{n=-\infty}^{-1} F(n) \cos\left(\frac{2\pi n}{p} x\right)$$

$$+\, i \sum_{n=1}^{\infty} F(n) \sin\left(\frac{2\pi n}{p} x\right) + i \sum_{n=-\infty}^{-1} F(n) \sin\left(\frac{2\pi n}{p} x\right)$$

$$F(x) = F(0) + \sum_{n=1}^{\infty} \left[F(n) + F(-n)\right] \cos\left(\frac{2\pi n}{p} x\right)$$

$$+ i \sum_{n=1}^{\infty} \left[F(n) - F(-n)\right] \sin\left(\frac{2\pi n}{p} x\right)$$

$$\boxed{= F(0) + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{2\pi n}{p} x\right) + b_n \sin\left(\frac{2\pi n}{p} x\right)\right]}$$

where $a_n = F(n) + F(-n)$ and $b_n = i\left(F(n) - F(-n)\right)$

⊕ compare with $f(z) = \sum_{n=-\infty}^{+\infty} F(n) z^n$ and $z = e^{\frac{2\pi i x}{p}}$

$f(z)$ is a function expanded about the circle $e^{2\pi i x/p}$

Now take $f(x)$ and apply:

$$\int_{-p/2}^{+p/2} f(x) \cos\left(\frac{2\pi m x}{p}\right) d\left(\frac{x}{p}\right) \quad \text{for integer } m.$$

F(0)

$$= \frac{1}{p} \int_{-p/2}^{p/2} dx \cos\left(\frac{2\pi m x}{p}\right) \left[\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n x}{p}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi n x}{p}\right)\right]$$

Consider different values of $m$,

If $m = 0$, $$\frac{1}{p} \int_{-p/2}^{+p/2} f(x)\, dx = \frac{a_0}{2p} \int_{-p/2}^{p/2} dx$$

$$+ \frac{1}{p} \int_{-p/2}^{p/2} dx \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n}{p} x\right) + \frac{1}{p} \int_{-p/2}^{p/2} dx \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi n}{p} x\right)$$

Since $\int_{-P/2}^{P/2} dx \cos\left(\frac{2\pi n}{P}x\right) = \int_{-P/2}^{P/2} dx \sin\left(\frac{2\pi n}{P}x\right)$

$m=0$:
$$\frac{1}{P}\int_{-P/2}^{P/2} f(x)\, dx = \frac{a_0}{2} + 0 + 0$$

average "DC" component

For $m \neq 0$, there are two cases:

$\boxed{n \neq m}$

$$\frac{1}{P}\int_{-P/2}^{P/2} dx\, a_n \cos\left(\frac{2\pi n x}{P}\right)\cos\left(\frac{2\pi m x}{P}\right) = 0 \quad \forall\, n \neq m$$

$$\frac{1}{P}\int_{-P/2}^{P/2} dx\, b_n \sin\left(\frac{2\pi n x}{P}\right)\cos\left(\frac{2\pi m x}{P}\right) = 0 \quad \forall\, n \neq m$$

$\boxed{n = m}$

$$\frac{1}{P}\int_{-P/2}^{P/2} dx\, a_m \cos^2\left(\frac{2\pi m x}{P}\right) = \frac{1}{P} a_m\left(\frac{P}{2}\right) = \frac{a_m}{2}$$

$$\frac{1}{P}\int_{-P/2}^{P/2} dx\, b_n \sin\left(\frac{2\pi n x}{P}\right)\cos\left(\frac{2\pi n x}{P}\right) = 0$$

$$\therefore \quad \boxed{a_m = \frac{2}{P}\int_{-P/2}^{P/2} dx\, f(x) \cos\left(\frac{2\pi m}{P}x\right)}$$

Similarly, if we integrated $\frac{1}{P}\int_{-P/2}^{P/2} f(x) \sin\left(\frac{2\pi m}{P}x\right) dx$,

we find

$$\boxed{b_m = \frac{2}{P}\int_{-P/2}^{P/2} f(x) \sin\left(\frac{2\pi m}{P}x\right) dx}$$

$\therefore$ For a p-periodic function $f(x)$,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n}{p}x\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi n}{p}x\right) \quad \text{, where}$$

the coefficients are found from

$$\boxed{a_0 = \frac{2}{p}\int_{-p/2}^{p/2} f(x)\,dx\,, \quad a_n = \frac{2}{p}\int_{-p/2}^{p/2} dx\, f(x) \cos\left(\frac{2\pi n}{p}x\right), \quad b_n = \frac{2}{p}\int_{-p/2}^{p/2} dx\, f(x) \sin\left(\frac{2\pi n}{p}x\right)}$$

⊛ Connect with continuous Fourier transform;

$$f(x) = \frac{1}{p}\int_{-p/2}^{p/2} f(x)\,dx + \sum_{n=1}^{\infty} \underbrace{\left(\frac{2}{p}\int_{-p/2}^{p/2} f(x')\cos\left(\frac{2\pi n x'}{p}\right)dx'\right)}_{a_n} \cos\left(\frac{2\pi n x}{p}\right)$$

$$+ \sum_{n=1}^{\infty} \underbrace{\left(\frac{2}{p}\int_{-p/2}^{p/2} f(x')\sin\left(\frac{2\pi n x'}{p}\right)dx'\right)}_{b_n} \sin\left(\frac{2\pi n x}{p}\right)$$

but $\cos y' \cos y + \sin y' \sin y = \cos(y' - y)$,

$$f(x) = \frac{1}{p}\int_{-p/2}^{p/2} f(x)\,dx + \sum_{n=1}^{\infty} \frac{2}{p}\int_{-p/2}^{p/2} f(x')\cos\left(\frac{2\pi n}{p}(x'-x')\right)dx'$$

let $p \to \infty$, $\quad \frac{2\pi}{p} \equiv \lambda_{n+1} - \lambda_n$ where $\lambda_n = \frac{2\pi n}{p}$

$$f(x) = \sum_{n=-\infty}^{+\infty} \frac{1}{p}\int_{-p/2}^{p/2} f(x')\cos\left(\frac{2\pi n}{p}(x'-x)\right)dx' \quad =$$

$$= \sum_{n=-\infty}^{+\infty} \frac{\lambda_{n+1}-\lambda_n}{2\pi} \int_{-P/2}^{P/2} f(x') \cos \lambda_n (x'-x)\, dx'$$

$$\xrightarrow[\text{Riemann Integral}]{P\to\infty} \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\lambda \int_{-\infty}^{+\infty} f(x')\, \underbrace{\cos\lambda(x'-x)}\, dx'$$

$$\boxed{\equiv \int_{-\infty}^{+\infty} dx' f(x')\, K(x'-x)}$$

Since $\sin\lambda(x'-x)$ is odd in $\lambda$,

$$\Rightarrow e^{i\lambda(x'-x)}$$

$$\Rightarrow f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\lambda \int_{-\infty}^{+\infty} f(x')\, e^{i\lambda(x'-x)}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\lambda\, e^{-i\lambda x} \underbrace{\int_{-\infty}^{+\infty} dx'\, e^{i\lambda x'} f(x')}_{\text{III}}$$

$F(\lambda)$, and if $\frac{\lambda}{2\pi} \equiv s$,

$$\left.\begin{aligned} f(x) &= \int_{-\infty}^{+\infty} ds\, e^{-2\pi i s x}\, F(s) \\ F(s) &= \int_{-\infty}^{+\infty} dx'\, e^{2\pi i s x'} f(x') \end{aligned}\right\}$$

$$f(\xi) = \int_{-\infty}^{+\infty} \frac{d\lambda}{\sqrt{2\pi}}\, e^{i\lambda\xi}\, F(\lambda)$$

$$F(\lambda) = \int_{-\infty}^{+\infty} \frac{d\xi}{\sqrt{2\pi}}\, e^{-i\lambda\xi}\, f(\xi)$$

Symmetric form

## Examples

$f(x) = e^{-a|x|}$

$$F(k) = \int_{-\infty}^{+\infty} e^{-a|x|} e^{-ikx}\, dx = \int_{-\infty}^{0} e^{-a|x|} e^{-ikx} dx + \int_{0}^{\infty} e^{-ax} e^{-ikx} dx$$

$$= \int_0^\infty e^{-ax} \cdot e^{ikx}\, dx + \int_0^\infty e^{-ax} e^{-ikx}\, dx$$

$$= 2\int_0^\infty e^{-ax} \cos kx\, dx = \left.\frac{e^{(ik-a)x}}{ik-a}\right|_0^\infty - \left.\frac{e^{-(ik+a)x}}{ik+a}\right|_0^\infty$$

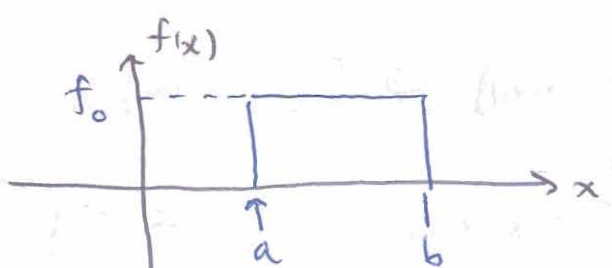
$$= \frac{1}{a+ik} + \frac{1}{a-ik} = \frac{2a}{a^2+k^2}$$

Lorentzian in $k$

Gaussian: $F(k) = \int_{-\infty}^{+\infty} e^{-ax^2 - ikx}\, dx$ Complete the square in $x$

$$F(k) = \int_{-\infty}^{+\infty} e^{-a(x+ik/2a)^2} e^{-k^2/4a} \underbrace{dx}_{d\left(x+\frac{ik}{2a}\right)} = e^{-k^2/4a}\sqrt{\frac{\pi}{a}}$$

Box function:




$$F(k) = \int_a^b f_0 e^{ikx}\, dx$$

$$= f_0 \left.\frac{e^{ikx}}{ik}\right|_{x=a}^{x=b} = f_0 \frac{e^{ikb} - e^{ika}}{ik}.$$ If $a=-b$, $F(k) = 2f_0 \frac{\sin kb}{k}$

Furthermore, if $f_0 = \frac{1}{2b}$

$$F(k) = \frac{\sin kb}{kb} \quad \text{as } b \to 0, \; f_0 \to \infty,$$

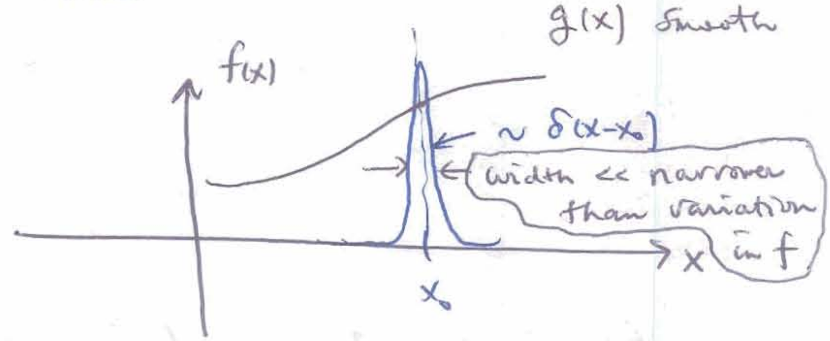
but $F(k) \to 1$

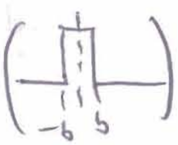
$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\lambda\, e^{-i\lambda x}\int_{-\infty}^{+\infty} dx'\, e^{i\lambda x'} f(x') = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dx' \underbrace{\left[\int_{-\infty}^{+\infty} d\lambda\, e^{i\lambda(x'-x)}\right]}_{\text{should pick out } x} f(x')$$

Delta-function, zero everywhere except where its argument $=0$,

$\delta(x)$, is a class of limiting functions

Normalized $\int_{-\infty}^{+\infty} \delta(x)\, dx \equiv 1$




$f(x) = \frac{1}{2b}$ 


$$\int_{x_0-\varepsilon}^{x_0+\varepsilon} f(x)\,\delta(x-x_0)\, dx = \int_{-\infty}^{+\infty} f(x)\,\delta(x-x_0)\, dx \equiv f(x_0)$$

$$\approx f(x_0)\int_{-\infty}^{+\infty} \delta(x-x_0)\, dx$$

$$\delta[a(x-x_0)] \Rightarrow \left(\frac{1}{a}\right)\delta(x-x_0)$$

$$\int_{-\infty}^{+\infty} \delta(a(x-x_0))\, g(x)\, dx \underset{ax=y}{=} \int_{-\infty}^{+\infty} \delta(y-y_0)\, g\left(\frac{y}{a}\right)\frac{dy}{a}$$

$$= \frac{1}{a}\, g(y_0/a) = \left(\frac{1}{a}\right) g(x_0)$$