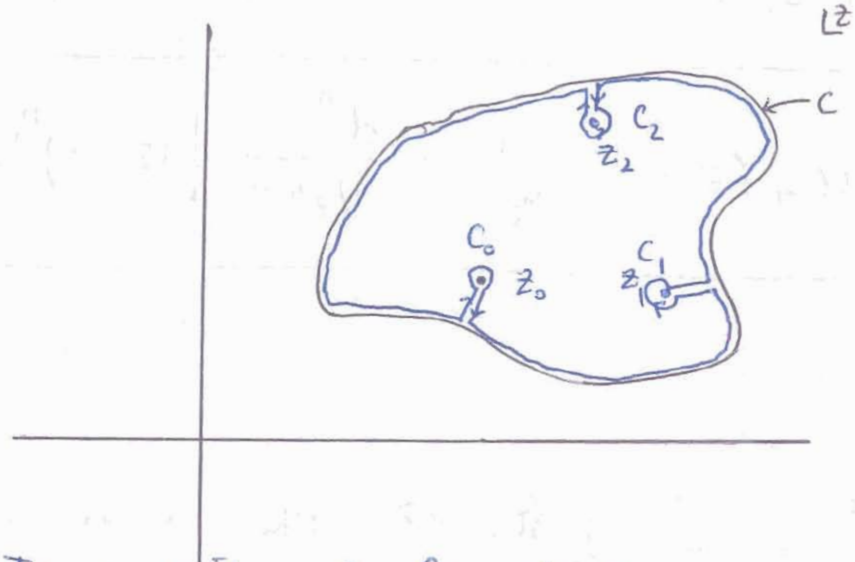
Contour Integration

$$\frac{1}{2\pi i}\oint_C f(z)\,dz = \text{Residue of } f(z) \text{ at pole } z_0 \text{ if } C \text{ surrounds } z_0$$

Residue of $f(z)$ is the coefficient of $\frac{1}{z-z_0}$ in a Laurent expansion

Multiple poles



each integral C_i contributes a residue of $f(z)$ at z_i

$$\boxed{\therefore \quad \frac{1}{2\pi i}\oint_C f(z)\,dz = \sum_i \text{Residues of } f(z) \text{ at poles } z_i}$$

Suppose $f(z)$ is analytic in and on C except a pole of order n at $z = z_0$. Then $f(z) \equiv \frac{g(z)}{(z-z_0)^n}$, where $g(z)$ is analytic everywhere.

$$\frac{1}{2\pi i}\oint f(z)\,dz = \frac{1}{2\pi i}\oint \frac{g(z)}{(z-z_0)^n}$$

$$\equiv \frac{1}{(n-1)!}\frac{\partial^{n-1}}{\partial z_0^{n-1}}\frac{1}{2\pi i}\oint \frac{g(z)}{(z-z_0)}\,dz \equiv \frac{1}{(n-1)!}\frac{\partial^{n-1}}{\partial z_0^{n-1}} f(z_0)$$

$$= \lim_{z\to z_0}\frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}\left[(z-z_0)^n f(z)\right]$$

$$\oint f(z)\,dz = 2\pi i \lim_{z\to z_0} \frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}\left((z-z_0)^n f(z)\right)$$

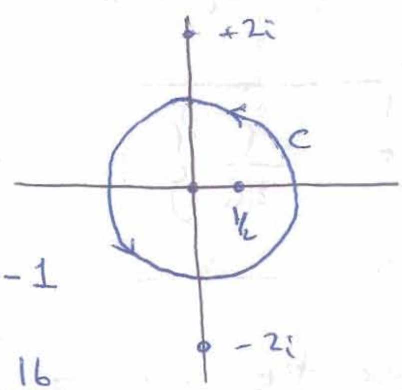
$$\equiv 2\pi i \left(\text{Residue of } f(z) \text{ at } z_0\right)$$

In general, residue of $f(z)$ at z_i, a pole of order n_i;

$$\boxed{a_{-1}(z_i) = \frac{1}{(n_i-1)!}\frac{d^{n_i-1}}{dz^{n_i-1}}\left[(z-z_i)^{n_i} f(z)\right]}$$

<u>Example</u> $\quad \frac{1}{2\pi i}\oint_C f(\zeta)\,d\zeta$ where C is unit circle and

$$f(\zeta) = \frac{1}{\zeta^2(\zeta-1/2)(\zeta^2+4)}$$



Only poles at $\zeta=0,\ 1/2$ matter:

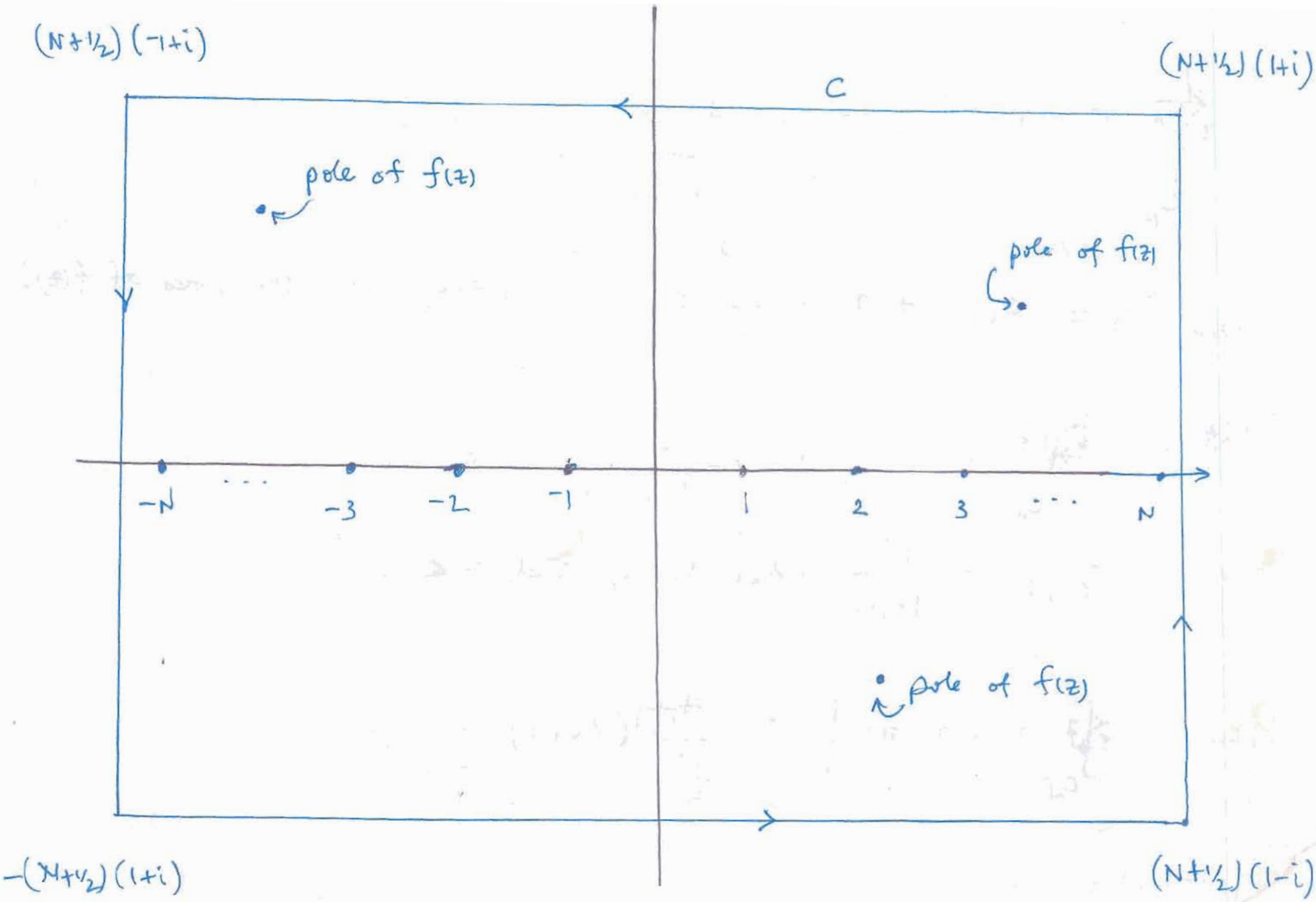
$\zeta=0$: $\quad n=2$, $\quad$ Residue $= \frac{\partial}{\partial\zeta}\left(\zeta^2 f(\zeta)\right)_{\zeta=0} = -1$

$\zeta=1/2$; $\ n=1$, $\quad$ Residue $= \frac{1}{(1/2)^2\left((1/2)^2+4\right)} = \frac{16}{17}$

$$\frac{1}{2\pi i}\oint f(\zeta)\,d\zeta = \left(\text{Residue at } \zeta=0\right) + \left(\text{Residue at } \zeta=1/2\right)$$

$$= -\frac{1}{17}$$

Summation of Series



choose $N \to \infty$ s.t. C encloses all poles of $f(z)$

poles of $\cot \pi z = \dfrac{\cos \pi z}{\sin \pi z}$ occur at $z = 0, \pm 1, \pm 2, \ldots$ (simple poles)

(why $\cos \pi z$ in numerator?)

Consider the function $\pi \cot \pi z \, f(z)$ at $z = 0, \pm 1, \pm 2, \ldots$

$$\lim_{z \to n} (z-n)\, \pi \cot \pi z \, f(z) = \lim_{z \to n} \pi \left(\frac{z-n}{\sin \pi z} \right) \cos \pi z \, f(z) = f(n)$$

$\therefore$ If $f(z)$ itself has no poles at $z = n$,

$$\oint_{C_N} \pi \cot \pi z \, f(z)\, dz = +\sum_{n=-N}^{+N} f(n) + S$$

where S = Sum of residues of $\pi \cot \pi z\, f(z)$ at the poles of $f(z)$!

what is $\oint_{C_N}$?

If $|f(z)| < \dfrac{M}{|z|^k}$ where $k > 1$, and $z \in C_N$

then $\left| \oint_{C_N} \pi \cot \pi z\, f(z) \right| \le \dfrac{A\pi M}{N^k}(8N+4) \xrightarrow[N\to\infty]{} 0$

$(|\cot \pi z| < A)$

$$\boxed{\therefore \sum_{n=-\infty}^{+\infty} f(n) = -S}$$

Example

$$\sum_{n=-\infty}^{+\infty} \frac{1}{n^2+a^2} = ?$$ consider $f(z) = \dfrac{1}{z^2+a^2}$ which has simple poles at $z = \pm ia$

$$\sum_{n=-\infty}^{+\infty} \frac{1}{n^2+a^2} = -\left(\text{Sum of residues of } \pi \cot f(z)\right) \text{ at } z = \pm ia$$

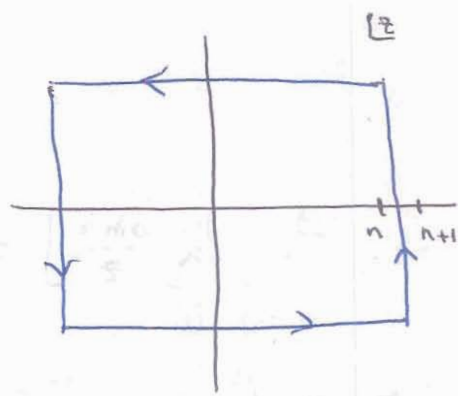
Residue of $\dfrac{\pi \cot \pi z}{z^2+a^2}$ at $z = +ia$ $= \dfrac{\pi \cot \pi(ia)}{2ia} = -\dfrac{\pi}{2a} \dfrac{\cosh(\pi a)}{\sinh(\pi a)}$

Residue of $\dfrac{\pi \cot \pi z}{z^2+a^2}$ at $z = -ia$ $= \dfrac{\pi \cot \pi(-ia)}{-2ia} = -\dfrac{\pi}{2a} \dfrac{\cosh(\pi a)}{\sinh(\pi a)}$

$$\sum_{n=-\infty}^{+\infty} \frac{1}{n^2+a^2} = \frac{\pi}{a} \coth(\pi a)$$

Sums and Product Relationships

Consider $I_n \equiv \dfrac{1}{2\pi i} \displaystyle\int_{C_n} \frac{d\zeta}{\zeta(\zeta - z)\sin\zeta}$ where



$$|\sin\zeta|^2 = \sin^2 x + \sinh^2 y$$

$\dfrac{1}{|\sin\zeta|} < 1$ on C_n and $I_n \to 0$ as $n \to \infty$

I_n = Sum of residues of integrand

Simple poles at $\zeta = n\pi$, z, double pole at $\zeta = 0$

$$\textcircled{1} = \frac{1}{z\sin z} - \frac{1}{z^2} + \sum_{n=1}^{\infty}\left[\frac{(-1)^n}{n\pi(n\pi-z)} + \frac{(-1)^n}{n\pi(n\pi+z)}\right]$$

$$\Rightarrow \quad \frac{1}{\sin z} = \frac{1}{z} - 2z\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2\pi^2-z^2}$$

if $z = \pi/2$,

$$1 = \frac{2}{\pi} - \pi\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2\pi^2-\pi^2/4} \Rightarrow \frac{\pi}{4} = \frac{1}{2} - 2\pi^2\sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2\pi^2-\pi^2}$$

$$= \frac{1}{2} - 2\sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2-1}$$

Similarly,

$$\boxed{\cot z = \frac{1}{z}} + 2z\sum_{n=1}^{\infty}\frac{1}{z^2-n^2\pi^2}$$

and note that $\displaystyle \frac{d}{dz}\left[\ln\frac{\sin z}{z}\right] = \frac{z}{\sin z}\frac{d}{dz}\left(\frac{\sin z}{z}\right)$

$$= \frac{z}{\sin z}\left(\frac{\cos z}{z} - \frac{\sin z}{z^2}\right) = \boxed{\cot z - \frac{1}{z}}$$

$$\therefore \frac{d}{dz}\left[\ln\frac{\sin z}{z}\right] = 2z\sum_{n=1}^{\infty}\frac{1}{z^2-n^2\pi^2} = \sum_{n=1}^{\infty}\frac{d}{dz}\left[\ln(z^2-n^2\pi^2)\right]$$

integrating $\int_0^z$, $\displaystyle \ln\frac{\sin z}{z} = \sum_{n=1}^{\infty}\left[\ln(z^2-n^2\pi^2) - \ln(-n^2\pi^2)\right]$

$$\underline{\frac{\sin z}{z}} = \exp\left[\sum_{n=1}^{\infty}\ln\left(\frac{n^2\pi^2-z^2}{n^2\pi^2}\right)\right] = \underline{\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2\pi^2}\right)}$$