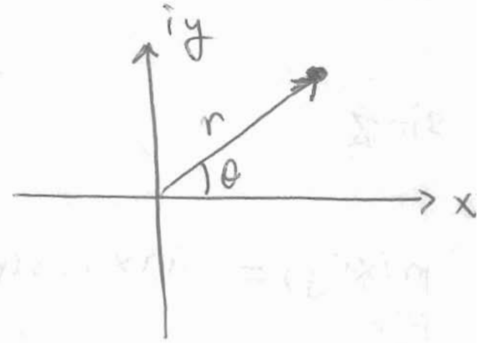
Complex Analysis

$z = x + iy$ $\qquad z = re^{i\theta}$

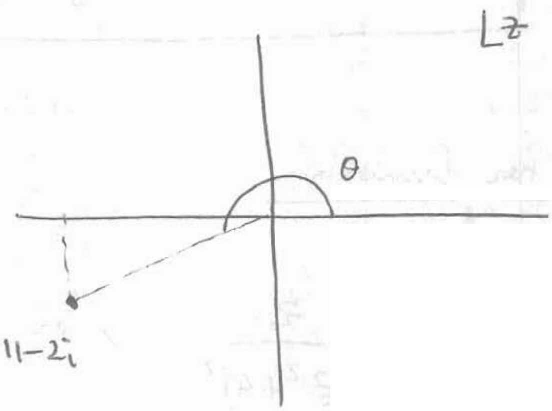


Roots

$$z^n = 1 = e^{2\pi i k} \qquad k \in \mathbb{Z}$$

if $\omega = e^{2\pi i/n}$, the n roots are

$$1, \omega, \omega^2, \ldots \omega^{n-1}$$



Example

Find roots of $z^3 = -11-2i$

$$-11-2i = \sqrt{121+4}\, e^{i\theta}$$

$$= \sqrt{125}\, e^{i\theta} \quad \text{where } \theta = \tan^{-1}\frac{2}{11} + \pi$$

$$z = \left(\sqrt{125}\right)^{1/3} \begin{pmatrix} e^{i\theta/3 \pm \frac{2\pi}{3}} \\ e^{i\theta/3} \end{pmatrix}$$

$$= \sqrt{5}\, e^{i\theta/3}, \ \sqrt{5}\, e^{i\theta/3 \pm \frac{2\pi i}{3}}$$

Functions of $z = x+iy$

example

$$f(z) = \sin z$$

$$= \sin(x+iy) = \sin x \cos(iy) + \cos x \sin(iy)$$

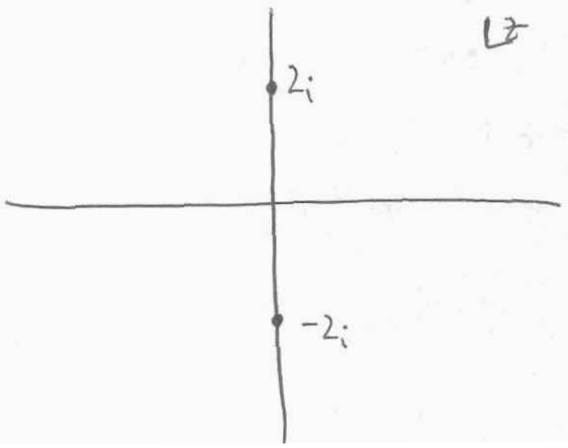
$$= \frac{e^{ix}-e^{-ix}}{2i}\,\frac{e^{i(iy)}+e^{-i(iy)}}{2} + \frac{e^{ix}+e^{-ix}}{2}\,\frac{e^{i(iy)}-e^{-i(iy)}}{2i}$$

$$= \sin x \cosh y + i \cos x \sinh y$$

$$\boxed{\sin z = \sin x \cosh y + i \cos x \sinh y}$$

Other functions

$$f(z) = \frac{z}{(z^2+4)^2}$$ has poles at $z = \pm 2i$, (second order poles)



pole of order 2

$$\lim_{z \to 2i} (z-2i)^2 f(z) \neq 0$$

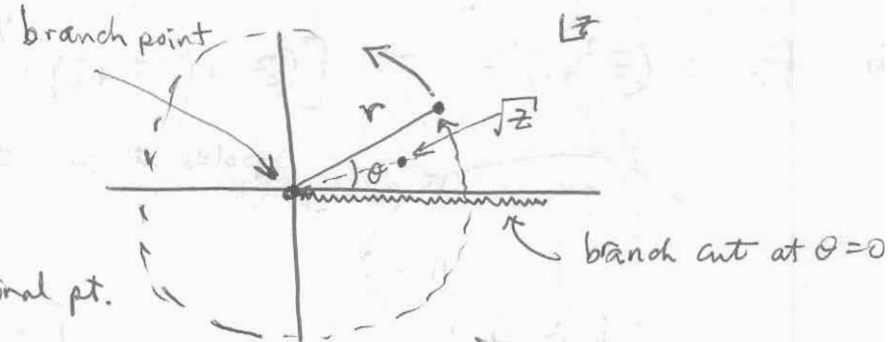
$$\boxed{\text{Isolated Singularity}}$$

$z = z_0$ is an isolated singularity if we can find $\delta > 0$ s.t.

$|z - z_0| = \delta$ enclosed no other singular point than z_0

$f(z) = \dfrac{3z-2}{(z-1)^2(z+1)(z-4)}$ poles of order 1 at $z=-1$, $z=4$
pole of order 2 at $z=1$

Branch cuts



* $f(z) = z^{1/2} = \sqrt{r}\, e^{i\theta/2}$

take $\theta \to \theta + 2\pi$

$z \to z$ back to original pt.

$f(z) \longrightarrow \sqrt{r}\, e^{i(\theta+2\pi)/2} = \sqrt{r}\, e^{i\theta/2} e^{i\pi} = -\sqrt{r}\, e^{i\theta/2}$

$\therefore\ f(z) \underset{\theta \to \theta+2\pi}{\longrightarrow} -f(z)$ Function changes sign.

If $\theta \to \theta + 4\pi$, $f(z) \to f(z)$

Thus, we restrict θ to say $0 < \theta \le 2\pi$ as one branch of $f(z)$

Another branch is $2\pi < \theta \le 4\pi$.

Only within each branch is $f(z)$ single valued.

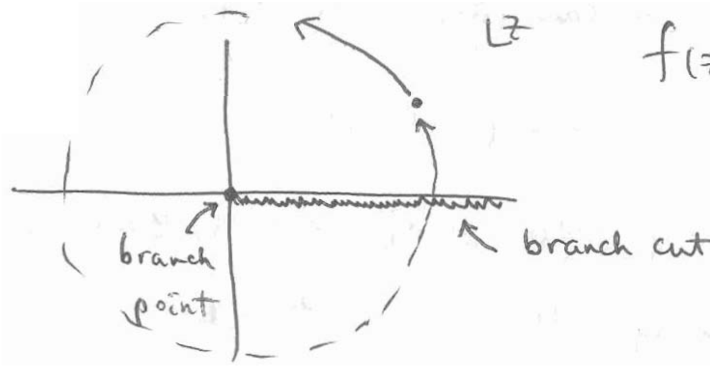
* $f(z) = \ln z = \ln(re^{i\theta}) = \ln r + i\theta$

$\theta \to \theta + 2\pi$, $f(z) \longrightarrow \ln r + i\theta + 2\pi i$

$\theta \to \theta + 4\pi$, $f(z) \longrightarrow \ln r + i\theta + 4\pi i$

$\Rightarrow$ infinitely many branches that don't come back to original value

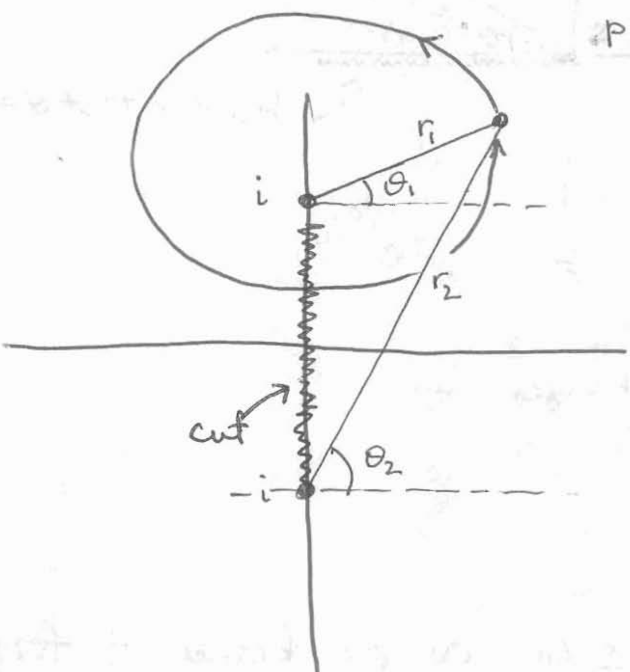
Define principal branch $0 < \theta \le 2\pi$, or $-\pi \le \theta < \pi$



$f(z) = \ln z$

⊛ $f(z) = (z^2+1)^{1/2} = \left[(z-i)(z+i)\right]^{1/2}$

poles at $z = \pm i$



let $z - i \equiv r_1 e^{i\theta_1}$

$z + i \equiv r_2 e^{i\theta_2}$

$$f(z) = \left[r_1 r_2 e^{i(\theta_1+\theta_2)}\right]^{1/2}$$

$$= \sqrt{r_1 r_2}\, e^{i(\theta_1+\theta_2)/2}$$

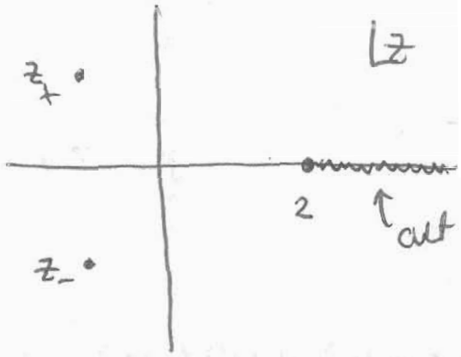
If $\theta_1 \to \theta_1 + 2\pi$, $f(z) \to \sqrt{r_1 r_2}\, e^{i(\theta_1+\theta_2)/2} e^{i\pi}$

$$= -f(z)$$

Encircling one pole goes through a cut

More complex function w/ cuts and singular points

$$f(z) = \frac{\ln(z-2)}{(z^2+2z+2)^4}$$



$$z_\pm = -1 \pm \tfrac{1}{2}\sqrt{4-8}$$

$= -1 \pm i$ are 4^{th} order poles

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* Isolated Singularities ; $f(z) = \frac{1}{(z-z_0)^n}$

* Branch cuts ; $f(z) = (z-z_0)^{1/2}$

* Essential Singularities — all else

e.g. $f(z) = e^{\frac{1}{z-2}}$ no Taylor or Laurent series

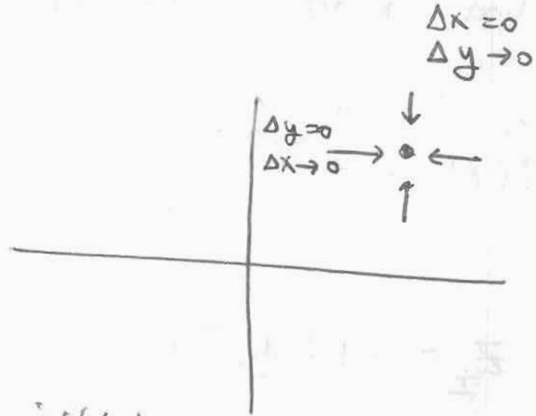
$\boxed{\text{Derivatives and Analytic Functions}}$

Consider a function that is analytic at z,
→ continuous and differentiable

$$f(z) = u(x,y) + i\,v(x,y)$$

Derivative of $f(z)$!

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} \equiv f'(z)$$



$$\lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) + iv(x, y+\Delta y) - u(x,y) - iv(x,y)}{i\Delta y}$$

$$= \frac{u(x, y+\Delta y) - u(x,y)}{i\Delta y} + i\frac{v(x, y+\Delta y) - v(x,y)}{i\Delta y}$$

$$= -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

If $f'(z)$ is single-valued, derivatives from both directions have to be the same:

$$\lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) + iV(x+\Delta x, y) - u(x,y) - iV(x,y)}{\Delta x}$$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

Equating real and imaginary parts,

$$\frac{\partial V}{\partial y} = \frac{\partial u}{\partial x} \qquad -\frac{\partial u}{\partial y} = \frac{\partial V}{\partial x}$$

$\boxed{\text{Cauchy - Riemann Conditions}}$

If second derivatives are defined,

$$\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\frac{\partial}{\partial y} u$$

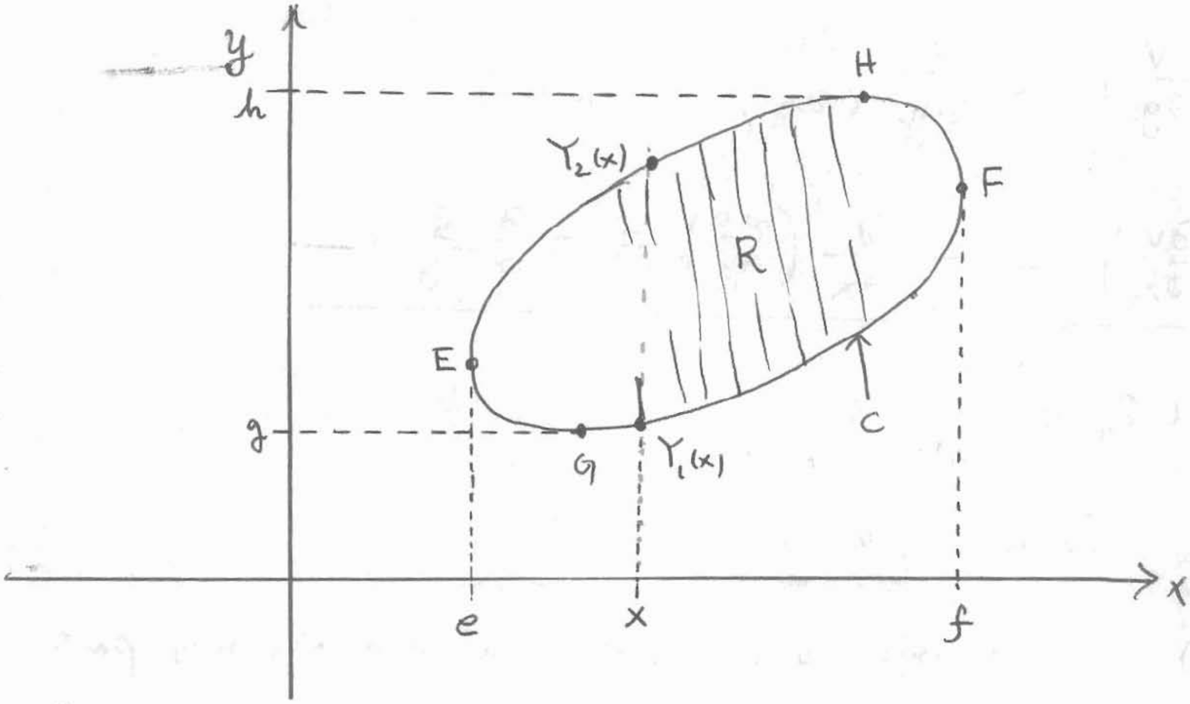
$$+ \quad \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right) = -\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = -\frac{\partial}{\partial x}\frac{\partial}{\partial y} u$$

$$\partial_x^2 v + \partial_y^2 v = 0$$

Similarly, $\partial_x^2 u + \partial_y^2 u = 0$

$\therefore$ $f(z)$ is harmonic, where both real and imaginary parts satisfy Laplaces Eqn.

Green's Theorem (2-D, not necessarily complex plane)



consider

$$\iint_R \frac{\partial P}{\partial y}\, dx\, dy \equiv \int_e^f dx \int_{Y_1(x)}^{Y_2(x)} dy\, \frac{\partial P}{\partial y}$$

Integrating, $\displaystyle \int_e^f dx\, P(x,y)\Big|_{y=Y_1(x)}^{y=Y_2(x)} = \int_e^f dx\,\Big[P(x,Y_2(x)) - P(x,Y_1(x))\Big]$

$$= -\int_e^f P(x,Y_1(x))\,dx - \int_f^e P(x,Y_2(x))\,dx$$

$$\equiv -\oint_C P\,dx$$

$$\therefore \quad \oint_C P\,dx = -\iint_R \frac{\partial P}{\partial y}\,dx\,dy$$

Now consider another function Q:

$$\iint_R \frac{\partial Q}{\partial x}\,dx\,dy = \int_g^h dy\,\left[\int_{x=X_1(y)}^{x=X_2(y)} dx\, \frac{\partial Q}{\partial x}\right] =$$

$$\int_g^h \left[Q(X_2(y),y) - Q(X_1(y),y) \right] dy$$

$$= \int_h^g Q(X_1(y),y)\,dy + \int_g^h Q(X_2(y),y)\,dy \equiv \oint Q\,dy$$

$$\therefore \boxed{\oint_C (P\,dx + Q\,dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy}$$

Green's Thm

Combine Green's Thm with Harmonic functions:

consider $\oint f(z)\,dz$ where $f(z)$ is analytic in and on C.

$$\oint_C f(z)\,dz = \oint_C (u+iv)(dx+i\,dy)$$

$$= \oint_C u\,dx - v\,dy + i \oint_C v\,dx + u\,dy$$

now identify $u \rightarrow P$ and $v \rightarrow -Q$ and $v \rightarrow P$, $u \rightarrow Q$

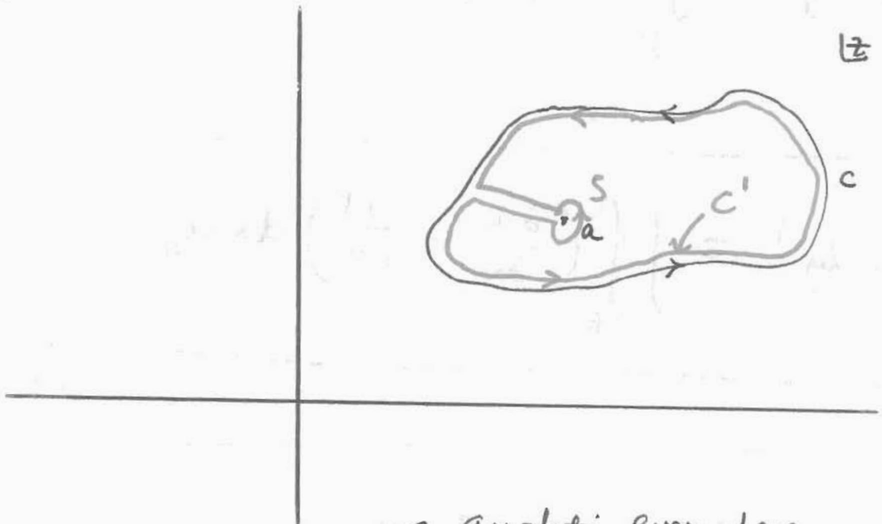
$$\oint_C f(z)\,dz = \iint_R \underbrace{\left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)}_{= 0} dx\,dy + i \iint_R \underbrace{\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right)}_{= 0} dx\,dy$$

from Cauchy Riemann conditions.

$$\therefore \boxed{\oint_C f(z)\,dz = 0} \Rightarrow \text{Cauchy Theorem}$$

Cauchy Integral Formula

Suppose the integrand has a pole inside C;



(pole at $z=a$)

What is $\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz = ?$ ← analytic everywhere

We can take the contour c' instead. From Cauchy's Thm,

$\frac{1}{2\pi i}\oint_{c'} \frac{f(z)}{z-a}\,dz = 0$. $\therefore$ the only contribution to $\oint_c \frac{f(z)}{z-a}\,dz$ comes from $\oint_S \frac{f(z)}{z-a}\,dz$

parameterized S by $z-a = \varepsilon e^{i\theta}$

$$\oint_S \frac{f(z)}{z-a}\,dz = \int_0^{2\pi} d\theta\, i\cancel{\varepsilon e^{i\theta}}\,\frac{f(a+\varepsilon e^{i\theta})}{\cancel{\varepsilon e^{i\theta}}}$$

$$= i\int_0^{2\pi} d\theta\, f(a+\varepsilon e^{i\theta}) \xrightarrow[\varepsilon\to 0]{} 2\pi i\, f(a)$$

$$\boxed{\therefore\ \frac{1}{2\pi i}\oint_c \frac{f(z)}{z-a}\,dz = f(a)}$$