The Method of Characteristics

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September 29, 2009

The Method of Characteristics is a general technique used to solve first order linear PDEs. However, one could always try this method on nonlinear equations if the "characteristics" (to be defined below) yield something tractible. The typical form to be considered is

$$a(x,t)\frac{\partial u(x,t)}{\partial t} + b(x,t)\frac{\partial u(x,t)}{\partial x} + f\left[u(x,t),x,t\right] = 0, \tag{1}$$

with initial condition $u(x,t=0) \equiv u_0(x)$. This form can be readily generalized to higher dimensions.

Note that $u(x,t)$ is a function of two independent variables x and t. The solution $u(x,t)$ defines a surface above the $x-t$ plane. In the Method of Characteristics, one tries to find a relationship between x and t such that along the curve $x(t)$, the equation for u simplifies and can be solved. Often, $u(x,t)$ is a constant along $x(t)$. However, $x(t)$ is only one trajectory that winds through $x-t$ space. However, if we can find $u(x,t)$ for each of the infinite number of non-crossing trajectories $x(t)$, then we have reconstructed $u(x,t)$ on the $x-t$ plane.

To go about this, we try to find general trajectories of both $x(s)$ and $t(s)$ as functions of a new coordinate s. Along the coordinate s,

$$\frac{\mathrm{d}u(x(s),t(s))}{\mathrm{d}s} = \frac{\mathrm{d}x}{\mathrm{d}s}\frac{\partial u}{\partial x} + \frac{\mathrm{d}t}{\mathrm{d}s}\frac{\partial u}{\partial t}. \tag{2}$$

Therefore, along s, the rate of change $\mathrm{d}u(x(s),t(s))/\mathrm{d}s$ is identical the first two terms of Eq. 1 provided

$$\frac{\mathrm{d}t}{\mathrm{d}s} = a(x(s),t(s)) \quad \text{and} \quad \frac{\mathrm{d}x}{\mathrm{d}s} = b(x(s),t(s)). \tag{3}$$

Often, $a(x,t) = 1$, and we can simply take $s = t$. Therefore, the only nontrivial trajectory is that of $x(t)$, and the ordinary differential equation that it solves is

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = b(x(t),t). \tag{4}$$

Therefore, along $x(t)$,

$$\frac{\mathrm{d}u(x(t),t)}{\mathrm{d}t} + f\left[u(x(t),t), x(t), t\right] = 0. \tag{5}$$

If this ODE is integrable, then we have some hope for an analytic solution.

To give a specific example of the procedure, let's also assume $f = 0$ and $b(x,t) = x$, so that

$$\frac{\mathrm{d}u(x(t),t)}{\mathrm{d}t} = 0 \tag{6}$$

and $x(t)$ determined by Eq. 4: $\mathrm{d}x/\mathrm{d}t = x$. Therefore, along the characteristic curves

$$x(t) = x(t=0)e^t \equiv x_0 e^t, \tag{7}$$

$u(x(t),t) =$ constant. Since we define the initial condition as $u(x,t=0) = u_0(x)$, different values x_0 of the position give the relationship

$$u(x_0 e^t, t) = u_0(x_0). \tag{8}$$

Upon redefining variables, our final solution that traces every value of x to a value of x_0 is

$$u(x,t) = u_0(xe^{-t}). \tag{9}$$

Let us now consider a more difficult problem. Assume that $u(x,y,t)$ is a function of three variables x, y, and t, and the PDE we wish to solve is

$$\partial u(x,y,t)\partial t + \mathbf{V}(x,y)\cdot\nabla u(x,y,t) = -kyu(x,y,t) \tag{10}$$

with initial condition $u(x,y,t=0) = \delta(x)\delta(y)$, and

$$V_x(x,y) = \sqrt{x(1-x)} \quad \text{and} \quad V_y(x,y) = \lambda(x-y). \tag{11}$$

Upon defining the characteristics $x(t)$ and $y(t)$, we find that if they satisfy

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = \sqrt{x(t)(1-x(t))} \quad \text{and} \quad \frac{\mathrm{d}y(t)}{\mathrm{d}t} = \lambda(x(t)-y(t)), \tag{12}$$

then

$$\frac{\mathrm{d}u(x(t),y(t),t)}{\mathrm{d}t} = -ky(t)u(x(t),y(t),t). \tag{13}$$

The trajectories that initially start at zero can be found explicitly:

$$\begin{aligned} x(t) &= \frac{1}{2}(1-\cos t) \\ y(t) &= \frac{1}{2} - \frac{\lambda^2\cos t + \lambda\sin t + e^{-\lambda t}}{2(\lambda^2+1)} < x(t). \end{aligned} \tag{14}$$

Along these trajectories we can find the solution

$$u(x, y, t) = \delta(x - x(t))\delta(y - y(t)) \exp\left[-k \int_0^t y(t')\mathrm{d}t'\right]. \tag{15}$$

This type of solution using Method of Characteristics commonly appears in simple stochastic problems, such as birth-death processes with immigration. If generating functions are used, the Master-equation for the generating function has a similar structure to the equations for u above.