Position & Size of boundary layers

* Can B.L. thickness $\delta \neq \varepsilon$?

$$\varepsilon y''(x) + a(x) y'(x) + b(x) y(x) = 0, \quad y(0) = A, \quad y(1) = B$$

$\Rightarrow$ if $a(x) > 0$, B.L. is on the ~~right~~ left
(for all x)

assume a rescaling $X = \frac{x}{\delta(\varepsilon)}$ where $\delta(\varepsilon) \neq \varepsilon$
i.e. $\delta \sim \varepsilon^{1/2}$ or $\delta \sim \varepsilon^2$

$$\left.\begin{aligned} \frac{dy}{dx} &= \frac{1}{\delta} \frac{dy_{in}}{dX} \\ \frac{d^2y}{dx^2} &= \frac{1}{\delta^2} \frac{d^2 y_{in}}{dX^2} \end{aligned}\right\} \quad \frac{\varepsilon}{\delta^2} \frac{d^2 y_{in}}{dX^2} + \frac{a(\delta X)}{\delta} \frac{dy_{in}}{dX} + b(\delta X) y_{in} = 0$$

Consider possibilities: $\delta(\varepsilon) \ll \varepsilon$, $\delta(\varepsilon) = \varepsilon$, and $\delta(\varepsilon) \gg \varepsilon$
$(\delta \sim \varepsilon^2)$ $(\delta \sim \sqrt{\varepsilon})$

* if $\delta(\varepsilon) \ll \varepsilon$, $\frac{\varepsilon}{\delta^2} \frac{d^2 y_{in}}{dX^2}$ is dominant term $\Rightarrow$ $y_{in} = A + BX$
$(y_{in}(0) = A)$
$y_{in}(X \to \infty) \sim Y_0(X \to \infty) \to \infty$ cannot match

* if $\delta(\varepsilon) \gg \varepsilon$, only one dominant term $a(0) \frac{dy_{in}}{dX} = 0$, $y_{in}(X) = A$
no match possible if $A \neq y_{out}(x \to 0)$

* if $\delta(\varepsilon) \sim \varepsilon$, there are two terms that balance:

$$\frac{d^2 y_{in}}{dX^2} + a(0) \frac{dy_{in}}{dX} \simeq 0 \Rightarrow$$ 2 integration const. and finite $X \to \infty$ limit

$$y_{in}(X) \simeq -\frac{z_0}{a(0)} e^{-a(0)X} + z_1$$ (decaying)

Consider $a(x)$ not always >0

not always >0

$$\varepsilon y''(x) - x^2 y'(x) - y = 0, \quad y(0) = y(1) = 1.$$

this problem will have two boundary layers (one near $x=0$, and one near $x=1$)

* Outer soln $\quad y_{out} = y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + \dots$

$O(\varepsilon^0)$: $\quad -x^2 y_0'(x) - y_0 = 0, \quad y_0(x) = C e^{-1/x}$ (no boundary conditions apply)

* Right boundary layer $\quad$ rescale $X = \frac{1-x}{\varepsilon}$ (when $x=1$, $X=0$)

assume $y_{in}^+ = Y_0^+ + \varepsilon Y_1^+ + \varepsilon^2 Y_2^+ + \dots$

$$O(\varepsilon^0): \quad \frac{d^2 Y_0^+}{dX^2} + \frac{dY_0^+}{dX} = 0 \Rightarrow Y_0^+(X) = A_0 + B_0 e^{-X}$$

use $y(x=0) = y_{in}^+(X=0) = Y_0^+(X=0) = 1$ boundary condition

$$A_0 + B_0 = 1$$

– matching

$$Ce^{1/x}(x \to 1) = Y_0^+(X \to \infty) = A_0$$

$\therefore \quad Ce = A_0$ and $B_0 = 1 - Ce$

C is still undetermined $\Rightarrow$ use left boundary condition

* Left boundary layer

rescale $Z = \frac{x}{\delta}$ where Z is the inner variable near $x=0$.
dominant balancing:

$$\overbrace{\frac{\varepsilon}{\delta^2} \underbrace{\frac{d^2 y_{in}^-(Z)}{dZ^2}}_{\frac{\varepsilon}{\delta^2} \sim \delta \Rightarrow \delta \sim \varepsilon^{1/3} \; (\ll y_{in}^- \text{ inconsistent})} - \underbrace{\delta Z^2 \frac{dy_{in}^-(Z)}{dZ}}_{\delta \sim 1 \text{ (no boundary layer)}}}^{\frac{\varepsilon}{\delta^2} \sim 1} - y_{in}^- = 0, \quad y_{in}^-(Z=0) = 1$$

$\Rightarrow$ The only consistent balance is $\delta \sim \sqrt{\varepsilon}$, so the boundary layer near $x=0$ has thickness $\sqrt{\varepsilon}$

set $\delta = \sqrt{\varepsilon}$;

$$\frac{d^2 y_{in}^-}{dZ^2} - y_{in}^- = \overbrace{\sqrt{\varepsilon}}^{\delta}\, Z^2 \frac{dy_{in}^-}{dZ}$$

expand $y_{in}^- = Y_0^-(Z) + \underbrace{\sqrt{\varepsilon}}_{\delta}\, Y_1^-(Z) + \dots \underbrace{\varepsilon}_{\delta^2} Y_2^- + \underbrace{\varepsilon^{3/2}}_{\delta^3} Y_3^- + \dots$

$O(\varepsilon^0)$: $\dfrac{d^2 Y_0^-(Z)}{dZ^2} - Y_0^-(Z) \approx 0, \quad Y_0^-(Z=0) = 1$

$$\Rightarrow Y_0^-(Z) = D_0 e^{+Z} + E_0 e^{-Z}, \quad D_0 + E_0 = 1$$

$D_0 = 0$ since $Y_0^-(Z \to \infty)$ should be finite to match outer soln.

$\therefore$ left side inner soln to $O(\varepsilon^0)$ is $Y_0^-(Z) \approx E_0 e^{-Z}$

– matching $Y_0^-(Z \to \infty) \to 0 \quad \therefore \quad C = 0$

i.e. $A_0 = C_e = 0, \; B_0 = 1,$ and $E_0 = 1$

matching at both ends $= 0$:

<u>to order ε^0:</u>

$$y_{unif} \simeq \underbrace{y_{out}}_{0} + Y_0^- + Y_0^+ - \underbrace{y^-_{match}}_{0} - \underbrace{y^+_{match}}_{0}$$

$$= 0 + e^{-Z} + e^{-X} - 0 - 0$$

$$= e^{-x/\sqrt{\varepsilon}} + e^{-(1-x)/\varepsilon} + \mathcal{O}(\sqrt{\varepsilon})$$