Continuous time Markov processes

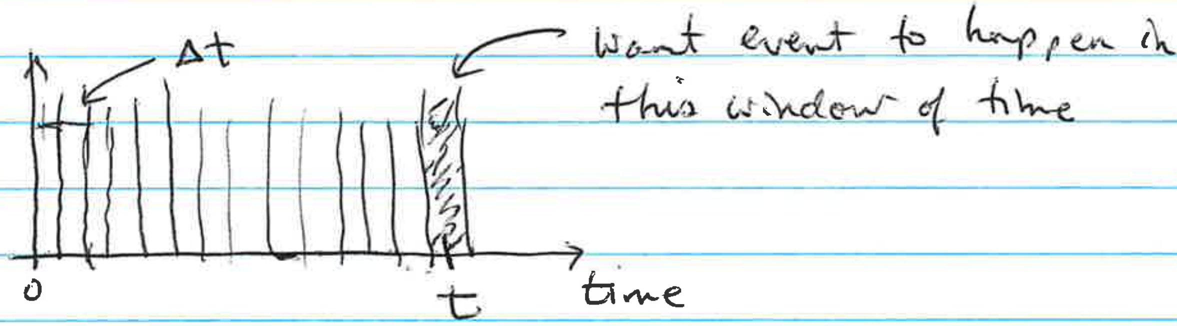
Instead of discrete steps, consider continuous time with

No memory:

transitions from one state to the next (like $n \to n+1$) occur instantly at a specific time. Transitions are "events", which have a probability intensity, or prob per time (rate)

$$P(n \to n+1, t)\, dt \equiv p\, dt$$

when does event occur?



in each increment, there is a probability $p\Delta t$ that event happens

$t = N\Delta t$, $1 - p\Delta t \equiv$ prob. that event did not happen in Δt

if there are N steps where event did not happen, prob. is $(1-p\Delta t)^N$

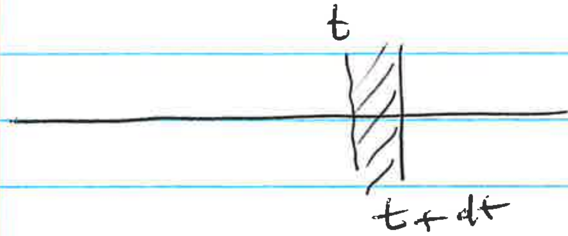
then, the event happens in the next Δt with probability $p\Delta t$

$\therefore$ $P(t, t+\Delta t)$ of event happening is

Prob. event happens between t and $t+\Delta t$ is

$$\left(1-p\frac{t}{N}\right)^N p\,\Delta t \xrightarrow[N\to\infty]{} pe^{-pt}\,dt$$

"waiting time" distribution $pe^{-pt} \equiv w(t)$



Suppose p depends on time $p = p_i$

$$\left(1-\frac{p_1\,dt}{N}\right)\left(1-\frac{p_2 t}{N}\right)\cdots\prod_{i=1}^{N}(1-p_i\,dt)$$

$$\ln(\Pi) = \sum_i \ln(1-p_i\,dt) \longrightarrow$$

generally,
waiting time distribution $w(t) = -\left.\frac{dS}{dt}\right|_t$

where $S(t)$ is the probably initial state survived (no transition) up to time t.

$$S(t) - S(t+\Delta t) = w(t)\,\Delta t$$

↓ → prob happened any time before t

$$[1-P_{ext}(t)] - [1-P_{ext}(t+dt)] = w(t)$$

$$\Rightarrow \underbrace{P_{ext}(t+\Delta t)}_{\text{prob of extinction up to time } t+\Delta t} - P_{ext}(t) = w(t)\,dt$$

general, not just Markov

$$\therefore\ w(t) = -\frac{dS}{dt}, \quad S(t) = e^{-pt}$$

What is probability that only one event occurred in $[0,T]$?

event happens at t_1, then no other events from t_1 to T, then integrate t_1 over $[0,T]$

prob no event up to t_1: $S(t_1) = e^{-pt_1}$

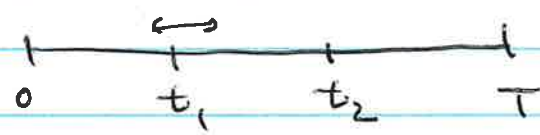
prob event at t_1, t_1+dt_1: $p\,dt_1$

prob of no event from t_1 to T: $e^{-p(T-t_1)}$

$$\therefore \int_0^T e^{-pt_1} e^{-p(T-t_1)} p\,dt_1$$

$= p e^{-pT} T$ is prob. that only one event occurred in $[0,T]$

now, extend to two events



prob that only one event occurred before t_2

$$p e^{-pt_2} t_2$$

$\times$ prob that second event occurs $p\,dt_2$

$\times$ prob that no other event occurs between t_2 and T:

$$e^{-p(T-t_2)}$$

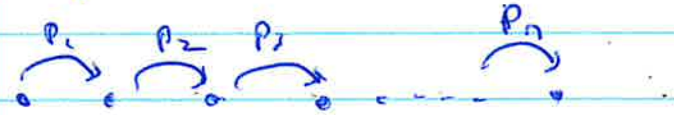
$\therefore$ prob that exactly two events occurred in $[0,T]$:

$$\int_0^T p t_2 e^{-pt_2} e^{-p(T-t_2)} p\,dt_2 =$$

$$= \rho^2 \int_0^T e^{-\rho T} t_2 \, dt_2 = \frac{(\rho T)^2}{2} e^{-\rho T}$$

for n events: $$\boxed{\text{Prob}\{n \text{ events in } (0,T)\} = \frac{(\rho T)^n}{n!} e^{-\rho T} \quad n = 0, 1, 2, \ldots \quad \text{Poisson distribution}}$$

Waiting time distribution for completing n steps



total time $T = t_1 + t_2 + t_3 + \ldots + t_n$ where each time is exponentially distributed.

Consider two events $\int_0^{t_2} \omega_1(t_1)\, \omega_2(t_2 - t_1)\, dt_1$ = prob density that time of second event is between t_2 and $t_2 + dt_2$

convolution; therefore we define $W_{12}(t_2)\, dt_2$ as prob second event occurs in $[t_2, t_2 + dt_2]$

Now, add a third event;

$$\int_0^{t_3} W_{12}(t_2)\, W_3(t_3 - t_2)\, dt_2$$ = prob density of 3rd event at time t_3

Another convolution

Use Laplace transforms $\mathcal{L}\left\{ \int_0^{t_2} \omega_1(t_1)\, \omega_2(t_2 - t_1)\, dt_1 \right\}$

$$= \mathcal{L}\{\omega_1\}\, \mathcal{L}\{\omega_2\}$$

$$\tilde{\omega}_1(s)\, \tilde{\omega}_2(s) = \tilde{\omega}_{12}(s)$$

$$\int_0^\infty \underbrace{(\rho_1 e^{-\rho_1 t})}_{\omega_1(t)} e^{-st}\, dt \equiv \tilde{\omega}_1(s)$$

$$= \frac{\rho_1}{\rho_1 + s} \;; \qquad \omega_{12}(s) = \frac{\rho_1 \rho_2}{(\rho_1 + s)(\rho_2 + s)}$$

$$\omega_{12}(t) = \frac{\rho_1 \rho_2}{\rho_2 - \rho_1} e^{-\rho_1 t} + \frac{\rho_1 \rho_2}{\rho_1 - \rho_2} e^{-\rho_2 t}$$

$$= \frac{\rho_1 \rho_2}{\rho_1 - \rho_2} \underbrace{\left(e^{-\rho_2 t} - e^{-\rho_1 t}\right)}_{} \xrightarrow[\rho_2 \to \rho_1]{}$$

$$e^{-\rho_2 t}\left(1 - e^{-(\rho_1 - \rho_2)t}\right)$$

$$e^{-\rho_2 t}\left[1 - (1 - (\rho_1 - \rho_2)t + \dots)\right]$$

$$\Rightarrow \frac{\rho_1 \rho_2}{\rho_1 - \rho_2} (\rho_1 - \rho_2)\, t\, e^{-\rho_2 t} \longrightarrow \rho_1^2\, t\, e^{-\rho_1 t}$$

For n steps $\quad \omega_{123\ldots n}(t) = \underbrace{\sum_{i=1}^{n} \left(\prod_{j \neq i} \frac{\rho_j}{\rho_j - \rho_i} \right) \rho_i e^{-\rho_i t}}_{}$

intermediate "hidden" states can be used to generate arbitrary waiting times

$$E[T] = \sum_{i=1}^{n} \frac{1}{p_i}$$

$$Var[T] = \sum_{i=1}^{n} \frac{1}{p_i^2}$$

For all $p_i = p$,

$$W_{12\ldots n}(t) = \frac{p^n t^{n-1} e^{-pt}}{(n-1)!}$$

Connect w/ Poisson $\quad P(n \text{ events before } T)$

$$= P(S_n \le T < S_{n+1})$$

where $S_n = t_1 + t_2 + \ldots + t_n$

exponentially distributed

and $S_{n+1} = t_1 + t_2 + \ldots t_n + t_{n+1} = S_n + t_{n+1}$

$$P(n \text{ event before } T) = P(S_n \le T < S_n + t_{n+1})$$

for some fixed $S_n = s$, we want $\underbrace{t_{n+1} > T - s}_{e^{-p\cdot(T-s)}}$

Integrate over all possible $s \le T$

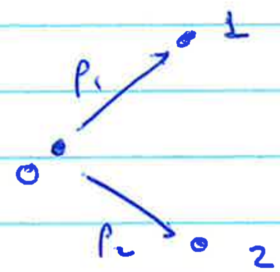
$$P(S_n \le T < S_{n+1}) = \int_0^T W_{12\ldots n}(s)\, e^{-p(T-s)}\, ds$$

$$P(n \text{ in time } T) = \int_0^T \frac{\rho^n s^{n-1}}{(n-1)!} e^{\cancel{-\rho s}} e^{-\rho(T\cancel{-s})}$$

$$= e^{-\rho T} \int_0^T \frac{\rho^n s^{n-1}}{(n-1)!} ds$$

$$= e^{-\rho T} \frac{(\rho T)^n}{n!} \checkmark$$

<u>Branching transitions</u>



rate of leaving is $\rho_1+\rho_2$

Survival in state 0 :

$$(1-\rho_1 dt-\rho_2 dt)^N \Rightarrow e^{-\overbrace{(\rho_1+\rho_2)}^{\text{total rate out}}t}$$

$$\text{total waiting time distribution} = -\frac{dS}{dt} = (\rho_1+\rho_2)e^{-(\rho_1+\rho_2)t}$$

What about conditional waiting time distributions?

$$W(t;1) = W(t|1)\,P(1)$$

$$\Downarrow$$

$$(1-\rho_1 dt-\rho_2 dt)^N \rho_1 = W(t|1)\,P(1), \quad P(1) = \int_0^\infty \rho_1 e^{-(\rho_1+\rho_2)t'} dt' = \frac{\rho_1}{\rho_1+\rho_2}$$

$$\rho_1 e^{-(\rho_1+\rho_2)t} = W(t|1)\left(\frac{\rho_1}{\rho_1+\rho_2}\right)$$

$$\Rightarrow W(t|1) = (\rho_1+\rho_2)e^{-(\rho_1+\rho_2)t}$$ Same as <u>unconditioned</u>

__Master equation__ discrete state n

$P(n,t\,|\,n_0,0)$ prob state is n at time t given start at state n_0 at time $t=0$

$$P(n,t+dt\,|\,n_0,0) = P(n,t\,|\,n_0,0)\left(1-\sum_m W_{mn}\,dt\right)$$

$$+\sum_m P(m\neq n,t\,|\,n_0,0)\,W_{nm}\,dt$$

$$\frac{P(n,t+dt\,|\,n_0,0)-P(n,t\,|\,n_0,0)}{dt}$$

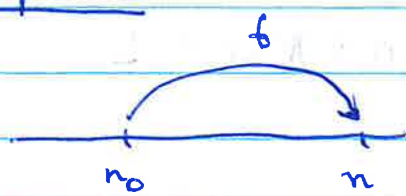
$$=\sum_m P(m,t\,|\,n_0,0)\,W_{nm}\cancel{dt}$$

$$-P(n,t\,|\,n_0,0)\left(\sum_m W_{mn}\right)\cancel{dt}$$

$$\Rightarrow\quad \frac{\partial P(n,t\,|\,n_0,0)}{\partial t} = \overbrace{\sum_m P(m,t\,|\,n_0,0)\,W_{nm}}^{\text{in rate from all states } m \text{ to state } n}$$

$$-\underbrace{P(n,t\,|\,n_0,0)\sum_m W_{mn}}_{\text{out rate to all states } m\neq n}$$

__Backward picture__



$$P(n,t\,|\,n_0,0) = P(n,t-\Delta t\,|\,n_0,0)$$

$$\times\left(1-\sum_m W_{mn_0}\Delta t\right)$$

$$+\sum P(n,t-\Delta t\,|\,k,0)\,W_{kn_0}\,\Delta t$$

$$P(n,t|n_0,0) - P(n,\underbrace{t-\Delta t}_{\text{"backward"}}|n_0,0) = -\left(\sum_m W_{mn_0}\right)\Delta t P(n,t-\Delta t|n_0,0) + \sum P(n,t-\Delta t|k,0)\, W_{kn_0}\Delta t$$

Shift time $t \to t+\Delta t$

$$\frac{P(n,t+\Delta t|n_0,0) - P(n,t|n_0,0)}{\Delta t} = \frac{\partial P(n,t|n_0,0)}{\partial t}$$

$$= \sum_k W_{kn_0} P(n,t|k,0) - P(n,t|n_0,0)\sum_m W_{mn_0}$$

RHS operates on initial position

Useful for considering residence in subsets of states:

we can sum over a set of final states of interest:

$$\sum_{n\in\mathcal{N}} P(n,t|n_0,0) \equiv S(t|n_0,0) \quad \text{survival within } \mathcal{N}$$

the sum can be taken over final states over the Backward equation

$$\frac{\partial S(t|n_0,0)}{\partial t} = \sum_k W_{kn_0} S(t|k,0) - S(t|n_0,0)\sum_m W_{mn_0}$$

Initial conditions $\quad S(0|n_0\in\mathcal{N},0) = 1$