Practice final

Questions 1-4 are short answer; Questions 5-12 are multiple choice (except for Q11)

- Q1. Find the row reduced form of

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 1 & 1 \\ 3 & 1 & 0 & 4 \\ 1 & -1 & -2 & -2 \end{pmatrix}$$

- Q2. Is the matrix

$$A = \begin{pmatrix} -2 & 1 & -1 \\ 0 & -2 & 0 \\ 0 & -2 & 1 \end{pmatrix}$$

invertible? Justify your reasoning.

- Q3. Let $A$ be the matrix

$$\begin{pmatrix} 2 & -1 & 1 & -4 \\ 1 & -1 & 1 & -2 \\ 3 & -1 & 1 & -6 \\ 1 & 0 & 0 & -2 \end{pmatrix}$$

and let $\vec{v}$ be the vector

$$\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}.$$

Compute the orthogonal projection of $\vec{v}$ onto the kernel $\ker(A)$ of $A$.

- Q4. Let $A$ be an $m \times n$ matrix with nullity zero, and let $v_1, \ldots, v_k$ be $k$ vectors in $\mathbf{R}^n$ which are linearly independent. Explain why the $k$ vectors $Av_1, \ldots, Av_k$ in $\mathbf{R}^m$ are also linearly independent.

- Q5. Let $A$ be the matrix

$$A = \begin{pmatrix} 2 & 3 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 2 \end{pmatrix}.$$

Then the *geometric* multiplicity of the eigenvalue $\lambda = 2$ is

- (a) 0
- (b) 1
- (c) 2
- (d) 3

- (e) 4

- Q6. Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the transformation that rotates the plane counter-clockwise by $\pi/6$ radians. Let $A$ be the matrix of $T$. Then the matrix of $A^{25}$ is
- (a) $\begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix}$.
- (b) $\begin{pmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{pmatrix}$.
- (c) $\begin{pmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{pmatrix}$.
- (d) $\begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}$.
- (e) $\begin{pmatrix} \sqrt{3}/2 & 1/2 \\ 1/2 & -\sqrt{3}/2 \end{pmatrix}$.
- (f) $\begin{pmatrix} -\sqrt{3}/2 & 1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix}$.

- Q7. Let $k$ be a real number. Then the system of equations

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 4 \\ x_1 + kx_2 + 4x_3 &= 6 \\ x_1 + 2x_2 + (k+2)x_3 &= 6 \end{aligned}$$

is consistent (i.e. has at least one solution) precisely when
- (a) $k = 1$.
- (b) $k \neq 1$.
- (c) $k = 2$.
- (d) $k \neq 2$.
- (e) $k = 1$ or $k = 2$.
- (f) $k \neq 1$ and $k \neq 2$.
- (g) The system is always consistent.
- (h) The system is never consistent.

- Q8. Let $V$ be the image of the matrix

$$\begin{pmatrix} 1 & 0 & 1 \\ 2 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then the dimensions of $V$ and of the orthogonal complement $V^\perp$ are
- (a) $\dim(V) = 4$ and $\dim(V^\perp) = 0$.
- (b) $\dim(V) = 3$ and $\dim(V^\perp) = 1$.
- (c) $\dim(V) = 3$ and $\dim(V^\perp) = 0$.
- (d) $\dim(V) = 2$ and $\dim(V^\perp) = 2$.
- (e) $\dim(V) = 2$ and $\dim(V^\perp) = 1$.

- (f) $\dim(V) = 1$ and $\dim(V^{\perp}) = 3$.
- (g) $\dim(V) = 1$ and $\dim(V^{\perp}) = 2$.
- (h) $\dim(V) = 0$ and $\dim(V^{\perp}) = 4$.
- (i) $\dim(V) = 0$ and $\dim(V^{\perp}) = 3$.

- Q9. Let $A, B$ be the matrices
$$A = \begin{pmatrix} 1 & 5 & 8 & 10 \\ 2 & 6 & 9 & 0 \\ 3 & 7 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{pmatrix}; \quad B = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 4 & 5 \end{pmatrix}.$$
Then the rank of $A^{-1}B$ is
- (a) 0
- (b) 1
- (c) 2
- (d) 3
- (e) 4
- (f) undefined ($A$ is not invertible)
  (Hint for Q9: you do *not* have to compute $A^{-1}$ or $A^{-1}B$ in order to solve this question, there is a quicker way.)

- Q10. Let $A$ be a $3 \times 5$ matrix. Then the statement $(\mathrm{im}(A))^{\perp} = \ker(A^T)$
- (a) is always true
- (b) is sometimes true
- (c) is never true

- Q11. What are the eigenvalues of
$$A = \begin{pmatrix} 3 & 0 & 1 \\ 2 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}?$$

- Q12. Let $A, B, C$ be the matrices
$$A = \begin{pmatrix} 4 & 0 \\ 2 & 5 \end{pmatrix}; B = \begin{pmatrix} 8 & -6 \\ 2 & 1 \end{pmatrix}; C = \begin{pmatrix} 2 & -4 \\ 4 & 2 \end{pmatrix}.$$
Then
- (a) $A$ is similar to both $B$ and $C$.
- (b) $A$ is similar to $B$ but is not similar to $C$.
- (c) $A$ is similar to $C$ but is not similar to $B$.
- (d) $A$ is not similar to either $B$ or $C$.