Math 33a(2), Solutions to the First Midterm

People did well on the midterm. There were a large number of perfect scores and the average was 17.4. (Anything below 10 and you might need to think about dropping.)

Q1: Find the rank of

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 7 & 9 \\ 10 & 11 & 12 \\ 1 & 0 & 0 \\ 0 & 2 & 2 \\ 1 & 1 & 0 \end{pmatrix}.$$

This question was graded my Jeremy Brandman.

Here it is essentially a matter of reducing down to rref using the elementary row operations. Most people had no trouble doing this, though a handful of people misstated the rank once they had it down to that form. The rank is actually 3.

One short cut which makes some sense with this problem is to just attack the last three rows. If you just reduce this "bottom half" of A you quickly end up with

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 7 & 9 \\ 10 & 11 & 12 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where apon those leading ones down the bottom can be used to annihilate everything on top, reducing to

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which then reduces down to

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

after we interchange first and fourth rows, the second and fifth, and then third and sixth.

This problem is easy enough without any clever shortcuts since we can just grind through with elementary row operation in a fairly small number of steps. The next problem, however, becomes much tougher if tackled head on.

Q2: T is the linear transformation corresponding to rotation by $\frac{\pi}{6}$ in a counterclockwise direction. We let A be the associated matrix and the problem asks for the matrix A^3.

Almost everyone charged directly at this problem by figuring out A and computing A^3 by laborious multiplication. There is a much better approach.

As remarked in class, matrix multiplication corresponds exactly to composition of linear transformations. Thus the matrix for the transformation

$$T \circ T \circ T$$

equals the matrix

$$A^3.$$

Given the definition of T it should be clear that T^3 is the transformation which rotates by an angle of

$$\frac{\pi}{6} + \frac{\pi}{6} + \frac{\pi}{6},$$

one $\pi/6$ for each time we rotate. Thus T^3 is rotation by the angle $3 \times \frac{\pi}{6} = \frac{\pi}{2}$. This has the matrix

$$(\, T^3(\vec{e_1}), T^3(\vec{e_2}) \,) = \begin{pmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \sin\frac{\pi}{2} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

which in turn, as indicated above, must equal A^3.

Note that this really is the right approach here. If you had instead been asked for A^{24} (and I considered putting this on the midterm), it would have been simply impossible to get the right answer by pure computation.

For this reason I was being a little bit harsher than usual towards people who tried the computational approach but made a mistake along the way. For instance, I was taking off at least one point, often two, when people made a mistake calculating out A^3. I took off two marks for getting the wrong answer as a result of miscalculating A. Two marks for not knowing $\cos\frac{\pi}{6}$ etc (though a few brave souls tried to figure it out from the appropriate isoceles triangle). And so on.

Q3: The inverse of

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 3 & 6 \end{pmatrix}.$$

This was graded by Lok Ming Lui.

One begins by row reducing

$$(\, A \quad I_3 \,)$$

and seeing how that leads out. (If we *do not* end up with I_3 on the left then A is *not* invertible. Otherwise the inverse is the matrix which ends up on the right.)

Most people did not seem to have too much trouble with row reduction. The matrix

$$(\, A \quad I_3 \,)$$

row reduces down to

$$\begin{pmatrix} 1 & 0 & 0 & 3 & -3 & 1 \\ 0 & 1 & 0 & -1 & 3 & -2 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{pmatrix},$$

thereby indicating to us that A is invertible and the inverse is

$$\begin{pmatrix} 3 & -3 & 1 \\ -1 & 3 & -2 \\ 0 & -1 & 1 \end{pmatrix}.$$

Q4: This was in effect a homework problem. We have the coefficient matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & k \\ 1 & 4 & k^2 \end{pmatrix}$$

and we want to know when

$$A\vec{x} = \begin{pmatrix} 5 \\ 10 \\ 21 \end{pmatrix}$$

has exactly one solution.

This is governed by the "important fact" which we keep coming back to. Since the coefficient matrix A is 3 by 3, we have $m = n = 3$. Thus, if the rank of $A < 3$ then there are always at infinitely many solutions or no solutions since rank is less than m (the number of unknowns, or columns). If $A = 3$ we are guaranteed a solution since the rank equals n (the number of rows, or equations), while there can be at most one solution since rank equals m.

In other words, just by looking at A we can say that there will be a unique solution if and only if A has rank 3. (And there was a homework problem which asked about this matrix A when it has rank 3.)

The next step then is to row reduce A. It comes down to

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & k-1 \\ 0 & 0 & (k-2)(k-1) \end{pmatrix}$$

and we then have to ask when this reduced form of A has rank 3. Clearly if $(k-2)(k-1)$ equals zero then we have rank less than three, whilst if $(k-2)(k-1) \neq 0$ we we can divide line three through by this quantity and then use that leading one to annihilate the column above, rapidly reducing down to I_3.

Thus the rank of the coefficient matrix is 3 (and there is a unique solution of the original system) if and only if

$$(k-2)(k-1) \neq 0,$$

which is $k \neq 2, 1$.