Mathematics 33A/2
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Midterm, Feb 2, 2005

Instructions: Try to do all five problems. The first three questions are worth 5 points each; the last two are multiple choice and are worth 3 points and 2 points respectively. Place your final answers in the boxes provided.

You may enter in a nickname if you want your midterm score posted.

Good luck!

Name: ____________________

Nickame: ____________________

Student ID: ____________________

Signature: ____________________

Problem 1 (5 points). __________

Problem 2 (5 points). __________

Problem 3 (5 points). __________

Problem 4 (3 points). __________

Problem 5 (2 points). __________

Total: ____________________

Problem 1. Find the inverse (if it exists) of the 3×3 matrix

$$A = \begin{pmatrix} 0 & 0 & 3 \\ 0 & 2 & 2 \\ 1 & 1 & 1 \end{pmatrix}.$$

The inverse exists, and is equal to

$$A^{-1} = \begin{pmatrix} 0 & -1/2 & 1 \\ -1/3 & 1/2 & 0 \\ 1/3 & 0 & 0 \end{pmatrix}.$$

This can be found by starting with the augmented matrix $(A|I)$ and row reducing the left matrix to equal I. Note that you will need to swap some rows to get A to be in the form I. The answer can be checked by showing that AA^{-1} or $A^{-1}A$ is equal to I.

Ans.

Problem 2. Suppose k is a real number such that the linear system of equations

$$\begin{array}{ccccccccc} & & y & + & 2z & + & 3w & = 4 \\ 5x & + & 6y & + & 7z & + & 8w & = 9 \\ 10x & + & 11y & + & 12z & + & 13w & = k \end{array}$$

has at least one solution for the unknowns x, y, z, w. What are the possible values of k? (Note: there are many ways to solve this problem. A little thought may be able to save you lots of computation).

The direct approach: if you double the second equation, you get

$$10x + 12y + 14z + 16w = 18,$$

and if you subtract the first equation from this you get

$$10x + 11y + 12z + 13w = 14.$$

Comparing this with the third equation, we see that k must equal 14. Conversely, if k *was* equal to 14, we can easily obtain a solution to the three equations, simply by finding a solution to the first two equations (because we have just shown that the first two equations imply the third; for instance, we can take $y = 4, z = 0, w = 0$ to satisfy the first equation, and then $x = -3$ to satisfy the second (and hence the third) equations.

Another way to proceed is to row reduce the augmented matrix

$$\left(\begin{array}{cccc|c} 0 & 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 & 9 \\ 10 & 11 & 12 & 13 & k \end{array}\right)$$

to get it into a form which is easier to solve. The row reduced echelon form is

$$\left(\begin{array}{cccc|c} 1 & 0 & -1 & -2 & -3 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & k-14 \end{array}\right).$$

Observe that the last row is inconsistent if $k - 14 \neq 0$, which shows that there are no solutions when k is not equal to 14. If $k = 14$ then the system is in row reduced echelon form and is consistent, and thus has a solution (in fact it has infinitely many solutions, because of the presence of free variables.

Ans. [ ]

Problem 3. Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ denote the linear transformation that is a rotation with the y axis as the axis of rotation, which rotates the positive x axis by $\frac{\pi}{4}$ radians (45 degrees) towards the positive z axis. Write down the matrix associated to T. (You do not need to simplify any of the sines and cosines that you may encounter).

Some elementary trigonometry shows that the vector $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ on the x axis rotates to $\begin{pmatrix} \cos(\frac{\pi}{4}) \\ 0 \\ \sin(\frac{\pi}{4}) \end{pmatrix}$, the vector $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ on the y axis rotates to $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ (note that the axis of rotation is not moved by the rotation), and the vector $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ on the z axis rotates to $\begin{pmatrix} -\sin(\frac{\pi}{4}) \\ 0 \\ \cos(\frac{\pi}{4}) \end{pmatrix}$. Thus the matrix associated to T is

$$\begin{pmatrix} \cos(\frac{\pi}{4}) & 0 & -\sin(\frac{\pi}{4}) \\ 0 & 1 & 0 \\ \sin(\frac{\pi}{4}) & 0 & \cos(\frac{\pi}{4}) \end{pmatrix}.$$

Ans.

Problem 4. Let A be an *invertible* 5×5 matrix, and let $T_A : \mathbf{R}^5 \to \mathbf{R}^5$ is the linear transformation associated to A. One of the following statements is *false*. Which one? (You do not need to explain your reasoning).

- (a) The rank of A must equal 5.
- (b) Every row of A must contain a leading 1.
- (c) For every vector b in $\mathbf{R}^5$, there must be exactly one solution to the equation $Ax = b$.
- (d) The row-reduced echelon form of A must be the identity matrix.
- (e) The image of T_A must equal $\mathbf{R}^5$.
- (f) The row-reduced echelon form of A must contain no free variables.
- (g) T_A must be both one-to-one and onto.
- (h) There must exist a 5×5 matrix B, such that $AB = BA = I$.

The answer is (b). All other statements are true. Note that while the *row-reduced echelon form* of an invertible matrix A must have a leading one in every row (in fact, it must be the identity matrix), the matrix A itself need not have a leading 1 in every row. (See for instance the invertible matrix in Q1).

Ans. [ ]

Problem 5. Let A be a 3×3 matrix, and suppose you know that the linear system of equations

$$A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

has no solutions. Then, the linear system of equations

$$A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

- (a) must have no solution.
- (b) must have exactly one solution.
- (c) must have infinitely many solutions.
- (d) We cannot conclude any of the above.

The answer is (c). Because the first linear system of equations has no solution, we know that A is not invertible. This means that the row reduced echelon form of A contains at least one zero row, and hence at least one free variable also. If one then considers the second linear system, which is homogeneous, we have a consistent row-reduced system of equations with at least one free variable, which thus has infinitely many solutions.

Ans.