Mathematics 33A/2
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Instructions: Try to do all five problems. The first three questions are worth 5 points each; the last two are multiple choice and are worth 3 points and 2 points respectively. Place your final answers in the boxes provided.

You may enter in a nickname if you want your midterm score posted.

Good luck!

Name: ____________________

Nickname: ____________________

Student ID: ____________________

Signature: ____________________

Problem 1 (5 points). ____________

Problem 2 (5 points). ____________

Problem 3 (5 points). ____________

Problem 4 (3 points). ____________

Problem 5 (2 points). ____________

Total: ____________________

Problem 1. Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ be the linear transformation given by the matrix

$$A = \begin{pmatrix} 1 & 3 & 5 \\ 1 & 2 & 4 \\ 1 & 1 & 1 \end{pmatrix}.$$

(a) Find a basis of vectors for the image $Im(T)$ of T.

The three columns here are linearly independent. This can be shown by a number of ways, for instance, by row-reducing the matrix all the way to the identity matrix, or at least reducing to a matrix for which the columns can be seen to be independent by inspection. Alternatively, one can see that the first column is non-redundant (it is non-zero), the second column is non-redundant (it is not a multiple of the first column), and the third column is non-redundant (it is not a linear combination of the first two columns, as can be seen by direct computation or row reduction). Thus the three columns

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 \\ 4 \\ 1 \end{pmatrix}$$

will form a basis for $Im(T)$. Actually since the three columns are linearly independent, they will span $\mathbf{R}^3$, and so in this case *any* three linearly independent vectors will form a basis of the image, such as the standard basis

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

In particular if one row-reduces an arbitrary matrix, it is not always the case that the columns of the row-reduced matrix will be a basis for the image.

Ans.

(b) Find a basis of vectors for the kernel $Ker(T)$ of T.

Since there are no redundant columns, the nullity here is zero, and so the empty basis $\emptyset$, consisting of no vectors whatsoever, will be a basis for the kernel $Ker(T)$, which in this case is the zero-dimensional space $\{0\}$ consisting only of the zero vector. Note that strictly speaking, the zero vector 0 is *not* a basis for the kernel; it is not linearly independent (the first vector is redundant). However as this is such a subtlety, I will accept the zero vector as a valid answer for this question, even if it is not strictly correct.

Ans.

Problem 2. Find the rank and nullity of the matrix

$$\begin{pmatrix} 1 & 6 & 1 & 9 \\ 2 & 7 & 2 & 0 \\ 3 & 0 & 3 & 0 \\ 4 & 8 & 4 & 10 \\ 5 & 0 & 5 & 11 \end{pmatrix}.$$

(Note: while it is possible to solve this problem by a full-blown row reduction, there are ways to solve the problem that require less computation.)

The first column is non-zero and hence non-redundant. The second column is not a scalar multiple of the first and is hence non-redundant. The third column is identical to the first and hence redundant. The fourth column cannot be a linear combination of the first three columns, for the following reason: the third column is redundant and thus can be ignored. By inspecting the third row of the matrix we see that one cannot write the fourth column using a non-zero multiple of the first column, and so one is left with writing the fourth column purely in terms of the second column. But this cannot be done either, as can be seen for instance by inspecting the second row of the matrix. Thus there are three non-redundant columns and one redundant one, so the rank is three and the nullity is one.

A number of people said that because the first and third columns were equal, that they were *both* redundant. This is incorrect. To be redundant, one has to be a linear combination of the *preceding* columns; this makes the third column redundant, but not the first.

Ans.

Problem 3. Let $B := (v_1, v_2, v_3)$ be the basis

$$v_1 := \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, v_2 := \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, v_3 := \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

of three vectors in $\mathbf{R}^3$. Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ be the linear transformation that swaps v_2 and v_3, but leaves v_1 unchanged (i.e. $T(v_1) = v_1$, $T(v_2) = v_3$, $T(v_3) = v_2$). Give a formula for what the transformation T does to a general vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ in $\mathbf{R}^3$; in other words, give a formula for $T\begin{pmatrix} x \\ y \\ z \end{pmatrix}$. (You do not need to simplify your formula; it is acceptable to have an answer which involves products and inverses of matrices, as long as all the matrices are specified explicitly. There are a number of ways to solve the problem; one way is to begin by first computing $[T]_B$).

First of all, the question is asking for what $T\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is. So the answer must be a vector, and it must involve x, y, z. An answer which is a matrix, or which does not involve x, y, z, will not be a correct answer and thus cannot earn full credit. Similarly, an answer which involves additional unspecified unknowns (e.g. c_1, c_2, c_3, or $a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, a_{32}, a_{33}$) is also not a correct answer and cannot earn full credit.

There are several ways to solve this question. One is the direct approach: since T is linear, it has the form

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

One can substitute this into the given information

$$T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; \quad T\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}; \quad T\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

and solve for a, b, c, d, e, f; the solution to this sub-problem is in fact

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$$

and hence the answer to the original problem is

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ y - z \end{pmatrix}.$$

One can then check this answer by inserting the original vectors v_1, v_2, v_3.

A slight variant of the above approach is to combine the three facts we know about T to obtain

$$T\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

and hence the matrix of T is given by

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^{-1}$$

and so another correct answer to the problem is given by

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^{-1}\begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

A third answer to the problem is to use the basis $B = (v_1, v_2, v_3)$. Observe that

$$[Tv_1]_B = [v_1]_B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; \quad [Tv_2]_B = [v_3]_B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}; \quad [Tv_3]_B = [v_2]_B = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

and hence

$$[T]_B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Since the basis matrix is given by

$$S = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

the matrix of T is then given by

$$S[T]_B S^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^{-1}$$

and so another correct form of the answer is

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^{-1}\begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Ans.

Problem 4. Let A be a 3×5 matrix (three rows and five columns). Then the nullity of A

- (a) must be equal to three.
- (b) must be equal to five.
- (c) must be equal to two.
- (d) can be any number between zero and two.
- (e) can be any number between two and five.
- (f) can be any number between zero and three.
- (g) can be any number between three and five.
- (h) can be any number between zero and five.

The answer is (e). There are five columns in $\mathbf{R}^3$, and so at most three of them are non-redundant. Thus anywhere between two to five columns can be redundant.

Ans.

Problem 5. Let v_1, v_2, v_3 be vectors in $\mathbf{R}^3$. Which of the following statements is *not* enough to imply that v_1, v_2, v_3 form a basis of $\mathbf{R}^3$?

- (a) The 3×3 matrix whose columns are given by v_1, v_2, v_3 is invertible.
- (b) Every vector in $\mathbf{R}^3$ is a linear combination of v_1, v_2, and v_3.
- (c) The vectors v_1, v_2, v_3 are all non-zero, and none of the vectors is parallel to any of the others.
- (d) None of the vectors v_1, v_2, v_3 are redundant.

The answer (c) is correct. Consider for instance three non-zero vectors which lie in a plane but such that none of the vectors are parallel to any other; a typical example is $(1, 0, 0), (0, 1, 0), (1, 1, 0)$, though this is of course not the only example. These vectors do not form a basis because they do not span all of $\mathbf{R}^3$.

Ans. | |
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