Math 33a(2), Solutions to the Final

Average on final: 54/80.

Note that there were many subtly different versions of the final, with various questions asked in different orders or slightly modified. The version I am going from below should be close enough to the others for the solutions to make sense for everyone.

Q1.

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 1 & 1 \\ 3 & 1 & 0 & 4 \\ 1 & -1 & -2 & -2 \end{pmatrix}.$$

rref(A)=I_4.

Of course absolutely everyone got this right.

Q2. With only one or two exceptions, practically everyone realized that the matrix

$$A = \begin{pmatrix} -2 & 1 & -1 \\ 0 & -2 & 0 \\ 0 & -2 & 1 \end{pmatrix}$$

is invertible. The main issue is that some people didn't pay much attention to the instruction that the answer must be *justified*. Reasonable justifications might be to point out that it has rank 3, or rref$(A) = I_3$, or to actually present A^{-1} and then verify $AA^{-1} = I_3$.

Q3.

$$A = \begin{pmatrix} 2 & -1 & 1 & -4 \\ 1 & -1 & 1 & -2 \\ 3 & -1 & 1 & -6 \\ 1 & 0 & 0 & -2 \end{pmatrix},$$

$$\vec{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}.$$

We want the projection of $\vec{v}$ to ker(A).

The kernel of A turns out to be the set of $\vec{x}$ with $x_2 = x_3, 2x_4 = x - 1$. This subspace in turn has an orthonormal basis

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Applying the usual formula for projection given an orthonormal basis we obtain

$$\begin{pmatrix} 12/5 \\ 5/2 \\ 5/2 \\ 6/5 \end{pmatrix}.$$

Q4. The main part of this problem was to prove that if A has nullity zero, then it sends linearly independent vectors to linearly independent vectors. Of course, nullity zero is another way of saying trivial kernal.

This question caused a lot of problems. The main difficulty is that many people would make some comments about kernal or mention rank plus nullity equal number of columns, or someother vaguely relevant looking fact, and then just leap to the conclusion. Here one needs a reasoned argument, which might convince a person not initially inclined to believe the proposition.

Here is how a correct proof might go:

"Let $\vec{v}_1, ...\vec{v}_\ell$ be linearly independent. Let $c_1, ..., c_\ell$ be scalars and suppose

$$c_1(A\vec{v}_1) + ... + c_\ell(A\vec{v}_\ell) = \vec{0}.$$

Since scalar multiplication commutes with matrix multiplication we have

$$A(c_1\vec{v}_1) + ... + A(c_\ell\vec{v}_\ell) = \vec{0}.$$

Then applying the distributivity properties of matrix multiplication and addition we have

$$A(c_1\vec{v}_1 + ... + c_\ell\vec{v}_\ell) = \vec{0}.$$

Then since nullity of A is zero this gives

$$c_1\vec{v}_1 + ... + c_\ell\vec{v}_\ell = \vec{0}.$$

Then we have $c_1 = 0, ..., c_\ell = 0$ by the assumption that $\vec{v}_1, ..., \vec{v}_\ell$ linearly independent."

Note we have implicitly used the characterization of linear independence in terms of no non-trivial combinations coming out to zero.

The Multiple Choice Problems

Q5.

$$A = \begin{pmatrix} 2 & 3 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 2 \end{pmatrix}.$$

The eigenvalue $\lambda = 2$ has *geometric multiplicity* 1 (but *algebraic multiplicity* 2).

Q6. For A the matrix inducing rotation of the plane by an angle of of $\pi/6$, we see that A^{25} induces rotation by $25\pi/6$, which mod 2π comes out to $\pi/6$ again. So the matrix is

$$\begin{pmatrix} (\sqrt{3})/2 & -1/2 \\ 1/2 & (\sqrt{3})/2 \end{pmatrix}.$$

Q7.

$$x_1 + 2x_2 + 3x_3 = 4,$$
$$x_1 + kx_2 + 4x_3 = 6,$$
$$x_1 + 2x_2 + (k+2)x_3 = 6,$$

is consistent exactly when $k \neq 1$. (Many people got part way into this and assumed we also need to require $k \neq 2$, but that is actually not true.)

Q8. V is the span of

$$\begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}.$$

The dimension of V is actually 2, and the dimension of $V^\perp$ is also 2. (This can be seen by e.g. crunching out the solutions for $\vec{x}\cdot\vec{v}_1 = 0, \vec{x}\cdot\vec{v}_2 = 0$, where $\vec{v}_1, \vec{v}_2$ is the basis of V consisting of the first two vectors above.)

There were lots of problems with this one. People seemed to think the dimension was 1, 3, 4, or even that $V^\perp$ was not a subspace.

Q9. Here we have 4 by 4 matrices A and B, where it was quickly obvious that the rank of A equals 4 and the rank of B is 3, but we need to compute rank $A^{-1}B$.

The point here is that A^{-1} must also be invertible and hence also have rank 4, and thus nullity zero. Then by the result from question 4 (among other things), rank $A^{-1}B$ must equal the rank of B.

I would expect that if you try to tackle this problem computationally, by figuring out A^{-1}, then $A^{-1}B$, then the rref, and so on, that the practical difficulties would be almost overwhelming.

Q10. It is *always* true that

$$(\text{im}A)^\perp = \ker(A^T).$$

(This was another question which gave a lot of trouble.)

Q11. The eigenvalues of

$$\begin{pmatrix} 3 & 0 & 1 \\ 2 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

are $4, 3, 1$. (E.g. expand the det of $A - \lambda I_3$ along the bottom row.)

Q.12

$$\begin{pmatrix} 4 & 0 \\ 2 & 5 \end{pmatrix}$$

is indeed similar to

$$\begin{pmatrix} 8 & -6 \\ 2 & 1 \end{pmatrix}$$

since they are both diagonalizable, and similar to the same diagonal matrix

$$\begin{pmatrix} 4 & 0 \\ 0 & 5 \end{pmatrix}.$$

However many people falsely thought it was also similar to

$$\begin{pmatrix} 2 & -4 \\ 4 & 2 \end{pmatrix}$$

since the determinants are the same. (While it is true that similar matrices have the same determinant, it does not always go the other way. Matrices with the same determinant may or may not be similar, depending on the circumstances.)