Mathematics 33A/2
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Instructions: Try to do all nine problems. The six short answer questions are worth 10 points each; the last three are multiple choice and are worth 5 points. The multiple choice answers require no additional explanation (and have no partial credit option), but some description of how you arrived at your answer is expected for each short answer question. In particular, a short answer with no justification provided whatsoever will only receive partial credit, even if correct. Place your final answers in the boxes provided; answers which do not appear in these boxes will only receive partial credit, even if correct.

You may enter in a nickname if you want your score posted.

Good luck!

Name: ______________________

Nickname: ______________________

Student ID: ______________________

Signature: ______________________

Problem 1 (10 points). ____________

Problem 2 (10 points). ____________

Problem 3 (10 points). ____________

Problem 4 (10 points). ____________

Problem 5 (10 points). ____________

Problem 6 (10 points). ____________

Problem 7 (5 points). ____________

Problem 8 (5 points). ____________

Problem 9 (5 points). ____________

Total: ______________________

Problem 1. Let A be the matrix

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

(a) Find an eigenbasis of vectors for A.

The eigenvalues of A are 1 and 4. The eigenspace E_1 is two-dimensional and consists of all the vectors $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ where $x + y + z = 0$, whereas E_4 is one-dimensional and consists of all the vectors of the form $\begin{pmatrix} x \\ x \\ x \end{pmatrix}$. So to pick an eigenbasis all one needs to do is to select one non-zero vector from E_4 and two independent vectors from E_3. There are many ways to do this, a typical one is

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

Many other solutions are, of course, possible.

(b) Find a diagonal matrix D and an invertible matrix S such that $A = SDS^{-1}$.

This is simply a matter of concatenating the previous answer and writing down the associated eigenvalues. For instance, with the above eigenbasis we would take

$$S = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}; \quad D = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Many other solutions are, of course, possible.

Problem 2. Let A be the matrix

$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \end{pmatrix}.$$

(a) Find a basis for $\ker(A)$. (Hint: it may be useful to first compute the nullity of A).

This matrix has two independent rows and thus has a nullity of two, so the kernel is two-dimensional. It is easiest to approach this problem by row reduction. One typical basis is

$$\begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -2 \\ 1 \end{pmatrix};$$

several other bases are possible. For instance, $\begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix}$ also lies in the kernel.

(b) Find an *orthonormal* basis for $\ker(A)$.

This is simply a matter of applying the Gram-Schmidt orthogonalization process to your preceding basis. A number of answers are possible, such as

$$\frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}; \frac{1}{\sqrt{30}}\begin{pmatrix} 2 \\ -1 \\ -4 \\ 3 \end{pmatrix}$$

or

$$\frac{1}{\sqrt{14}}\begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix}; \frac{1}{\sqrt{70}}\begin{pmatrix} -1 \\ -2 \\ 7 \\ -4 \end{pmatrix}.$$

Your answer should, of course be (a) orthonormal, (b) lie in the kernel.

Problem 3. Let A be the lower-triangular matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Find an orthogonal matrix Q and an upper-triangular matrix R such that $A = QR$.

This is a matter of applying the Gram-Schmidt orthogonalization process to the vectors

$$v_1 := \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}; \quad v_2 := \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}; \quad v_3 := \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}; \quad v_4 := \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

The correct result of this process is

$$u_1 = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}; \quad u_2 = \frac{1}{\sqrt{12}}\begin{pmatrix} -3 \\ 1 \\ 1 \\ 1 \end{pmatrix}; \quad u_3 = \frac{1}{\sqrt{6}}\begin{pmatrix} 0 \\ -2 \\ 1 \\ 1 \end{pmatrix}; \quad u_4 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}.$$

Note that there are a number of ways to write this answer because the square roots can be rearranged a little bit. Many of you got the first or second vectors right, but gave incorrect answers for the third or fourth; one way to detect a problem is to verify at the end that u_1, u_2, u_3, u_4 are indeed orthonormal. The associated matrices Q, R are

$$Q = \begin{pmatrix} 1/2 & -3/\sqrt{12} & 0 & 0 \\ 1/2 & 1/\sqrt{12} & -2/\sqrt{6} & 0 \\ 1/2 & 1/\sqrt{12} & 1/\sqrt{6} & -1/\sqrt{2} \\ 1/2 & 1/\sqrt{12} & 1/\sqrt{6} & 1/\sqrt{2} \end{pmatrix}; \quad R = \begin{pmatrix} 4/2 & 3/2 & 2/2 & 1/2 \\ 0 & 3/\sqrt{12} & 2/\sqrt{12} & 1/\sqrt{12} \\ 0 & 0 & 2/\sqrt{6} & 1/\sqrt{6} \\ 0 & 0 & 0 & 1/\sqrt{2} \end{pmatrix}.$$

Again, there are various ways to simplify or rewrite the above answers.

Problem 4. Provide an example of a 4×4 *orthogonal* matrix whose last column is

$$\begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix}.$$

(Hint: if you are alert, you may notice a very easy way to solve this problem.)

One quick way is simply to rearrange the matrix Q from the previous problem, thus for instance

$$\begin{pmatrix} -3/\sqrt{12} & 0 & 0 & 1/2 \\ 1/\sqrt{12} & -2/\sqrt{6} & 0 & 1/2 \\ 1/\sqrt{12} & 1/\sqrt{6} & -1/\sqrt{2} & 1/2 \\ 1/\sqrt{12} & 1/\sqrt{6} & 1/\sqrt{2} & 1/2 \end{pmatrix}$$

is a valid answer. Alternatively one may just use trial and error. There are several answers, including

$$\begin{pmatrix} -1/2 & -1/2 & -1/2 & 1/2 \\ -1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 & 1/2 \end{pmatrix}, \begin{pmatrix} 0 & 1/\sqrt{2} & -1/2 & 1/2 \\ 0 & -1/\sqrt{2} & -1/2 & 1/2 \\ 1/\sqrt{2} & 0 & 1/2 & 1/2 \\ -1/\sqrt{2} & 0 & 1/2 & 1/2 \end{pmatrix}$$

and a number of variations on these themes. A number of people noted correctly that orthogonal matrices must have determinant ± 1, but this does NOT imply that any matrix with determinant ± 1 is necessarily orthogonal. This led to a number of incorrect answers. Note that one can always check one's work at the end by seeing if the columns are indeed orthogonal to each other and also of unit length.

Problem 5. Let a and b be real numbers, and let A be the matrix

$$\begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix}.$$

For which values of a, b is A diagonalizable?

The eigenvalues of this matrix are 1 and b. If $b \neq 1$ then the eigenvalues are distinct, and A is automatically diagonalizable. Now suppose $b = 1$, then, 1 has algebraic multiplicity 2. The geometric multiplicity can be seen to equal 2 when $a = 0$ and to equal 1 otherwise, so when $b = 1$ we have diagonalizability if and only if $a = 0$. Thus the answer is that A is diagonal when $b \neq 1$, or when $b = 1$ and $a = 0$.

Problem 6. For which values of k is the matrix

$$\begin{pmatrix} 1 & 0 & 0 & k \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

invertible? (Row reduction will work, but is messy and prone to error; there is a faster way).

The easiest way to do this is by determinants. The above matrix has determinant $1-k$, so one has invertibility precisely when $k \neq 1$.

Problem 7. Let A be the 5×3 matrix

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 5 \\ 1 & 5 & 6 \end{pmatrix}.$$

Then the orthogonal complement of $Im(A)$ has dimension

- (a) zero.
- (b) one.
- (c) two.
- (d) three.
- (e) four.
- (f) five.
- (g) six.

$Im(A)$ is two dimensional (two independent columns) in $\mathbf{R}^5$, so its orthogonal complement has dimension three. Thus the correct answer is (d).

Problem 8. Let V be the subspace of $\mathbf{R}^5$ defined by

$$V := \{(0, x, 0, y, z) : x, y, z \in \mathbf{R}\}.$$

Let v be the vector $v := (1, 2, 3, 4, 5)$. Then the orthogonal projection of v onto V is

- (a) $(0, 0, 0, 0, 0)$
- (b) $(1, 2, 3, 0, 0)$
- (c) $(0, 0, 0, 4, 5)$
- (d) $(0, 1, 0, 2, 3)$
- (e) $(0, 2, 0, 4, 5)$
- (f) $(1, 0, 3, 0, 0)$
- (g) $(1, 2, 3, 4, 5)$

The answer is (e). This can be seen by taking an orthonormal basis of V such as $(0, 1, 0, 0, 0), (0, 0, 0, 1, 0), (0, 0, 0, 0, 1)$, or by inspection, since $(0, 2, 0, 4, 5)$ lies in V and $v - (0, 2, 0, 4, 5)$ is orthogonal to V.

Problem 9. If x, y, z solves the linear system of equations

$$\begin{array}{llll} ax & +by & +cz & = d \\ ex & +fy & +gz & = h \\ ix & +jy & +kz & = l \end{array}$$

and the determinant D defined by

$$D := \det \begin{pmatrix} a & b & c \\ e & f & g \\ i & j & k \end{pmatrix}$$

is non-zero, then y is equal to

- (a) $\frac{1}{D} \det \begin{pmatrix} d & b & c \\ h & f & g \\ l & j & k \end{pmatrix}$.
- (b) $\frac{1}{D} \det \begin{pmatrix} a & d & c \\ e & h & g \\ i & l & k \end{pmatrix}$.
- (c) $\frac{1}{D} \det \begin{pmatrix} d & h & l \\ e & f & g \\ i & j & k \end{pmatrix}$.
- (d) $\frac{1}{D} \det \begin{pmatrix} a & b & c \\ d & h & l \\ i & j & k \end{pmatrix}$.
- (e) $-\frac{1}{D} \det \begin{pmatrix} a & d & c \\ e & h & g \\ i & l & k \end{pmatrix}$.
- (f) $-\frac{1}{D} \det \begin{pmatrix} a & b & c \\ d & h & l \\ i & j & k \end{pmatrix}$.

Hint: if you don't recall the answer, try some examples.

The answer is (b) (Cramer's rule).