Mathematics 133
Terence Tao
Midterm, Feb 11, 2003

Instructions: Try to do all five problems; they are all of equal value. There is plenty of working space, and a blank page at the end.

You may enter in a nickname if you want your midterm score posted.

Good luck!

Name: ____________________

Nickame: ____________________

Student ID: ____________________

Signature: ____________________

Problem 1. __________

Problem 2. __________

Problem 3. __________

Problem 4. __________

Problem 5. __________

Total: ____________________

Problem 1. Let $f : \mathbf{R} \to \mathbf{R}$ be the function defined by setting $f(x) = 1$ when $0 \leq x \leq \pi$, and $f(x) = 0$ when $-\pi < x < 0$, and then extending periodically by 2π (thus for instance $f(x) = 1$ when $2\pi \leq x \leq 3\pi$); such a function is sometimes called a *square wave*. Compute the Fourier coefficients $\hat{f}(n)$ of f (Caution: the case $n = 0$ may have to be treated separately. You may find the identity $e^{\pi i n} = (-1)^n$ useful.). Using Parseval's identity, conclude that

$$\sum_{n \in \mathbf{Z}: n \text{ odd}} \frac{1}{n^2} = \ldots + \frac{1}{(-5)^2} + \frac{1}{(-3)}^2 + \frac{1}{(-1)}^2 + \frac{1}{1}^2 + \frac{1}{3}^2 + \frac{1}{5}^2 + \ldots = \frac{\pi^2}{4}.$$

First we compute the zeroth Fourier mode $\hat{f}(0)$:

$$\hat{f}(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\ dx = \frac{1}{2\pi} \int_0^{\pi} 1\ dx = \frac{1}{2\pi}\pi.$$

Next, we compute the non-zero Fourier modes $\hat{f}(n)$, $n \neq 0$:

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\ dx = \frac{1}{2\pi} \int_0^{\pi} e^{-inx}\ dx = \frac{1}{2\pi} \frac{e^{-inx}}{-in}|_0^{\pi} = \frac{1 - e^{-in\pi}}{2\pi i n} = \frac{1 - (-1)^n}{2\pi i n}.$$

In particular we see that $\hat{f}(n) = 0$ for even n, and $\hat{f}(n) = \frac{1}{\pi i n}$ for odd n. In other words

$$f(x) \sim \frac{1}{2} + \sum_{n \in \mathbf{Z}: n \text{ odd}} \frac{1}{\pi i n} e^{inx}.$$

By Parseval's identity we thus have

$$\|f\|^2 = \frac{1}{4} + \sum_{n \in \mathbf{Z}: n \text{ odd}} |\frac{1}{\pi i n}|^2 = \frac{1}{4} + \frac{1}{\pi^2} \sum_{n \in \mathbf{Z}: n \text{ odd}} \frac{1}{n^2}.$$

But we have

$$\|f\|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2\ dx = \frac{1}{2\pi} \int_0^{\pi} 1\ dx = \frac{1}{2\pi}\pi = \frac{1}{2}$$

and hence after a little algebra we obtain

$$\sum_{n \in \mathbf{Z}: n \text{ odd}} \frac{1}{n^2} = \frac{\pi^2}{4}$$

as desired.

Problem 2. Let f be a 2π-periodic function which is continuous and differentiable on the interval $-\pi < x < \pi$, but has jump discontinuities at $x = -\pi$ and $x = \pi$. Suppose also that f and its derivative remain bounded on the interval $-\pi < x < \pi$; more precisely, suppose that there exists a constant $M > 0$ such that $|f(x)| \leq M$ and $|f'(x)| \leq M$ for all $-\pi < x < \pi$.

(a) Establish the bound

$$|\hat{f}(n)| \leq \frac{2M}{|n|} \text{ for all } n \neq 0.$$

(Hint: integrate by parts).

Let $f(\pi^-)$ be the left limit of f at π; observe that $|f(\pi^-)| \leq M$ since $|f(x)| \leq M$ for all $-\pi < x < \pi$. Similarly if we let $f(-\pi^+)$ be the right limit of f at $-\pi$ we have $|f(-\pi^+)| \leq M$. Now we integrate by parts to obtain

$$\begin{aligned}
|\hat{f}(n)| &= |\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx| \\
&= \frac{1}{2\pi}\left|f(x)\frac{e^{-inx}}{-in}|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} f'(x)\frac{e^{-inx}}{-in}\,dx\right| \\
&= \frac{1}{2\pi}\left|f(\pi^-)\frac{e^{-in\pi}}{-in} - f(-\pi^+)\frac{e^{in\pi}}{-in} - \int_{-\pi}^{\pi} f'(x)\frac{e^{-inx}}{-in}\,dx\right| \\
&\leq \frac{1}{2\pi}(M\frac{1}{n} + M\frac{1}{N} + |\int_{-\pi}^{\pi} f'(x)\frac{e^{-inx}}{-in}\,dx|) \\
&\leq \frac{1}{2\pi}(\frac{2M}{n} + \int_{-\pi}^{\pi} M\frac{1}{n}\,dx) \\
&\leq \frac{1}{2\pi}(\frac{2M}{n} + \frac{2\pi M}{n}) \\
&\leq \frac{1}{2\pi}\frac{4\pi M}{n} \\
&= \frac{2M}{n}
\end{aligned}$$

as desired.

(b) Explain briefly why the series $\sum_{n=-\infty}^{\infty} |\hat{f}(n)|$ cannot be absolutely convergent. (Hint: Do not try to use (a). Instead, you can use the fact that a uniformly convergent series of continuous functions must be continuous (or, what amounts to much the same thing, you can use the Weierstrass M-test)).

Suppose for contradiction that $\sum_{n=-\infty}^{\infty} |\hat{f}(n)|$ is absolutely convergent. Then the Fourier series $\sum_{n=-\infty}^{\infty} \hat{f}(n)e^{inx}$ converges absolutely and uniformly to f (by the Weierstrass M-test). But since each of the terms in this Fourier series are continuous, this means that f is itself continuous, a contradiction.

Note, by the way, that (a) and (b) together give a (very indirect!) proof that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ must be divergent. One can also use Q1 and Q2(b) to obtain a similar result.

Problem 3. Let f be a Riemann integrable function, let $N \geq 0$ be a non-negative integer, and let $\sigma_N(f) = \frac{S_0(f)+\ldots+S_{N-1}(f)}{N}$ be the N^{th} Fejér sum (or Cesáro sum), as defined in page 53 of the textbook. Prove that

$$\sigma_N(f)(x) = \sum_{n=-N}^{N} (1 - \frac{|n|}{N})\hat{f}(n)e^{inx}$$

for all x. (For comparison, recall that $S_N f(x) = \sum_{n=-N}^{N} \hat{f}(n)e^{inx}$).

We have

$$\sigma_N(f)(x) = \frac{S_0(f)(x) + \ldots + S_{N-1}(f)(x)}{N} = \frac{1}{N}\sum_{k=0}^{N-1} S_k(f)(x) = \frac{1}{N}\sum_{k=0}^{N-1}\sum_{n=-k}^{k} \hat{f}(n)e^{inx}.$$

Observe that the only values of n which can appear are when $|n| \leq k \leq N-1$. Observe that the term $\hat{f}(n)e^{inx}$ appears in all the inner sums for which $N-1 \geq k \geq |n|$, and does not appear in any of the others. For each fixed n, the number of k between $N-1$ and $|n|$ is $N-|n|$, thus we have

$$\sigma_N(f)(x) = \frac{1}{N}\sum_{n:|n|\leq N-1} (N-|n|)\hat{f}(n)e^{inx}.$$

Note we can freely add the $|n| = N$ terms to this sum since they are zero. The claim now follows by moving the $\frac{1}{N}$ factor inside.

Problem 4. A function f is called *even* if one has $f(x) = f(-x)$ for all x, and is called *odd* if one has $f(x) = -f(-x)$ for all x. Show that if f and g are Riemann-integrable 2π-periodic functions which are both odd, then $f * g$ is even. (In other words, the convolution of two odd functions is an even function).

Proof A (sketch): First observe that a function f is even if and only if $\hat{f}(n) = \hat{f}(-n)$ for all n, and is odd if and only if $\hat{f}(n) = -\hat{f}(-n)$ for all n; these facts can be shown by comparing the Fourier coefficients of $f(x)$ and $f(-x)$. The claim will then follow from the identity $\widehat{f * g}(n) = \hat{f}(n)\hat{g}(n)$.

Proof B: For any x, we compute

$$f * g(-x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) g(-x-y)\ dy = \frac{1}{2\pi} \int_{-\pi}^{\pi} (-f(-y))(-g(x+y))\ dy.$$

Making the change of variables $z = -y$ (and remembering to change the limits of integration) we obtain

$$f * g(-x) = \frac{1}{2\pi} \int_{\pi}^{-\pi} (-f(z))(-g(x-z))\ (-dz).$$

Reversing the limits of integration and cancelling all the signs, we obtain

$$f * g(-x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(z) g(x-z)\ dz = f * g(x)$$

and the claim follows.

Problem 5. Let u, v be elements of a complex inner product space V. Show that

$$\langle u, v\rangle = \frac{\|u+v\|^2 - \|u-v\|^2 + i\|u+iv\|^2 - \|u-iv\|^2}{4}.$$

(This identity, sometimes called the *complex parallelogram identity*, shows that if one is given the magnitudes of all the vectors in a vector space, one can reconstruct the inner product also.)

Observe that

$$\|u+v\|^2 = \langle u+v, u+v\rangle = \langle u, u\rangle + \langle u, v\rangle + \langle v, u\rangle + \langle v, v\rangle.$$

Replacing v by $-v$ we obtain

$$\|u-v\|^2 = \langle u, u\rangle - \langle u, v\rangle - \langle v, u\rangle + \langle v, v\rangle.$$

Subtracting these two equations we obtain

$$\|u+v\|^2 - \|u-v\|^2 = 2\langle u, v\rangle + 2\langle v, u\rangle.$$

Replacing v by iv we obtain

$$\|u+iv\|^2 - \|u-iv\|^2 = 2\langle u, iv\rangle + 2\langle iv, u\rangle;$$

pulling the factors of i out and then multiplying this by i we obtain

$$i\|u+iv\|^2 - i\|u-iv\|^2 = 2\langle u, v\rangle - 2\langle v, u\rangle.$$

Adding this to a previous equation we obtain

$$\|u+v\|^2 - \|u-v\|^2 + i\|u+iv\|^2 - \|u-iv\|^2 = 4\langle u, v\rangle$$

and the claim follows.
