Mathematics 133
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Instructions: Do all nine problems; they are all of equal value. There is plenty of working space, and a blank page at the end.

You may enter in a nickname if you want your final score posted. Good luck!

Name: ____________________

Nickame: ____________________

Student ID: ____________________

Signature: ____________________

Problem 1 (10 points).	
Problem 2 (10 points).	
Problem 3 (10 points).	
Problem 4 (10 points).	
Problem 5 (10 points).	
Problem 6 (10 points).	
Problem 7 (10 points).	
Problem 8 (10 points).	
Problem 9 (10 points).	
Total (90 points):	

Problem 1. Let $N \geq 1$ be an integer, and let Ω be a subset of $\mathbf{Z}/N\mathbf{Z}$ which consists of precisely A elements. Let $f : \mathbf{Z}/N\mathbf{Z} \to \mathbf{C}$ be the function defined by setting $f(x) := 1$ when $x \in \Omega$ and $f(x) := 0$ when $x \notin \Omega$. (This function is sometimes called the *indicator function* or *characteristic function* of Ω).

(a) Show that $\hat{f}(0) = A/N$, and that $|\hat{f}(\xi)| \leq A/N$ for all other values of ξ.

We have

$$\hat{f}(0) = \frac{1}{N} \sum_{x \in \mathbf{Z}/N\mathbf{Z}} f(x) e^{2\pi i 0 x/N} = \frac{1}{N} \sum_{x \in \mathbf{Z}/N\mathbf{Z}} f(x) = \frac{1}{N} \sum_{x \in \Omega} 1 = \frac{A}{N}$$

and

$$|\hat{f}(\xi)| = |\frac{1}{N} \sum_{x \in \mathbf{Z}/N\mathbf{Z}} f(x) e^{2\pi i \xi x/N}| \leq \frac{1}{N} \sum_{x \in \mathbf{Z}/N\mathbf{Z}} |f(x)||e^{2\pi i \xi x/N}| = \frac{1}{N} \sum_{x \in \Omega} 1 = \frac{A}{N}$$

where we have used the triangle inequality and the fact that $e^{2\pi i \xi x/N}$ has magnitude 1.

(b) Show that $\sum_{\xi \in \mathbf{Z}/N\mathbf{Z}} |\hat{f}(\xi)|^2 = A/N$.

By Plancherel's theorem we have

$$\sum_{\xi \in \mathbf{Z}/N\mathbf{Z}} |\hat{f}(\xi)|^2 = \frac{1}{N} \sum_{x \in \mathbf{Z}/N\mathbf{Z}} |f(x)|^2 = \frac{1}{N} \sum_{x \in \Omega} 1^2 = \frac{A}{N}.$$

Problem 2. Let $f \in \mathcal{S}(\mathbf{R})$ be a Schwartz function with the property that $f(x) = 0$ for $|x| \geq 1/2$ (in other words, f is supported on the interval $[-1/2, 1/2]$). Prove the identity

$$\hat{f}(\xi) = \sum_{n=-\infty}^{\infty} \hat{f}(n) \frac{\sin(\pi(n-\xi))}{\pi(n-\xi)}$$

for every non-integer frequency $\xi \in \mathbf{R} \backslash \mathbf{Z}$. (Hint: start with the Poisson summation formula $\sum_{n=-\infty}^{\infty} f(x+n) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi i n x}$ applied to any $-1/2 < x < 1/2$, multiply both sides by $e^{-2\pi i x \xi}$, and integrate both sides in x from $-1/2$ to $1/2$. You may freely interchange integrals and summations without justification). The above identity is known as the *Nyquist-Shannon sampling theorem*, and plays an important role in information theory and in signal processing.

Starting with

$$\sum_{n=-\infty}^{\infty} f(x+n) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi i n x}$$

and multiplying by $e^{-2\pi i x \xi}$, we obtain

$$\sum_{n=-\infty}^{\infty} f(x+n) e^{-2\pi i x \xi} = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi i n (x-\xi)}$$

and then integrating x from $-1/2$ to $1/2$, we obtain

$$\int_{-1/2}^{1/2} \sum_{n=-\infty}^{\infty} f(x+n) e^{-2\pi i x \xi}\, dx = \int_{-1/2}^{1/2} \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi i x (n-\xi)}\, dx.$$

Interchanging the integral and sum, we obtain

$$\sum_{n=-\infty}^{\infty} \int_{-1/2}^{1/2} f(x+n) e^{-2\pi i x \xi}\, dx = \sum_{n=-\infty}^{\infty} \int_{-1/2}^{1/2} \hat{f}(n) e^{2\pi i x (n-\xi)}\, dx.$$

Observe that in the integral $\int_{-1/2}^{1/2} f(x+n) e^{-2\pi i x \xi}\, dx$, the quantity $x+n$ ranges between $n - 1/2$ and $n + 1/2$. But f is only non-zero between $-1/2$ and $1/2$. Thus for non-zero n, the integral $\int_{-1/2}^{1/2} f(x+n) e^{-2\pi i x \xi}\, dx$ is zero. Thus the left-hand side is

$$\int_{-1/2}^{1/2} f(x) e^{-2\pi i x \xi}\, dx.$$

Again, since f vanishes outside of $[-1/2, 1/2]$, this is the same as

$$\int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi}\, dx = \hat{f}(\xi).$$

Now we turn to the right-hand side. Observe that

$$\int_{-1/2}^{1/2} \hat{f}(n)e^{2\pi ix(n-\xi)}\, dx = \hat{f}(n)\frac{e^{2\pi ix(n-\xi)}}{2\pi i(n-\xi)}\Big|_{-1/2}^{1/2} = \hat{f}(n)\frac{2i\sin\pi(n-\xi)}{2\pi i(n-\xi)} = \hat{f}(n)\frac{\sin\pi(n-1)}{\pi(n-\xi)}.$$

Thus the left-hand side is $\sum_{n=-\infty}^{\infty}\hat{f}(n)\frac{\sin\pi(n-1)}{\pi(n-\xi)}$. The claim follows.

Problem 3. Let $u(t,x) : \mathbf{R} \times \mathbf{R} \to \mathbf{C}$ be a solution to the *one-dimensional wave equation*

$$\frac{\partial^2 u}{\partial t^2}(t,x) = \frac{\partial^2 u}{\partial x^2}(t,x).$$

Suppose that for each time t, $u(t,x)$ is a Schwartz function. Define the spatial Fourier transform $\hat{u}(t,\xi)$ of u at time t and frequency ξ in the usual manner by the formula

$$\hat{u}(t,\xi) = \int_{\mathbf{R}} u(t,x) e^{-2\pi i x \xi}\, dx.$$

(a) Show that $\hat{u}(t,\xi)$ must be of the form

$$\hat{u}(t,\xi) = F(\xi) e^{2\pi i t \xi} + G(\xi) e^{-2\pi i t \xi}$$

for some functions $F : \mathbf{R} \to \mathbf{C}$ and $G : \mathbf{R} \to \mathbf{C}$. (You may freely interchange time derivatives with the Fourier transform without justification, and assume as much differentiability in time as you wish).

Taking Fourier transforms in space of the wave equation, we obtain

$$\frac{\partial^2 \hat{u}}{\partial t^2}(t,\xi) = (2\pi i \xi)^2 \hat{u}(t,\xi)$$

or in other words

$$\frac{\partial^2}{\partial t^2}\hat{u} + 4\pi^2 \xi^2 \hat{u} = 0.$$

The characteristic polynomial $c^2 + 4\pi^2\xi^2 = 0$ has simple roots at $c = +2\pi i \xi$ and $c = -2\pi i \xi$, so for each ξ, the solution to the above equation is of the form

$$\hat{u}(t,\xi) = F(\xi) e^{2\pi i t \xi} + G(\xi) e^{-2\pi i t \xi}$$

for some F and G. (To actually solve for F and G, we would need to know both the initial position $u(0,x)$ and initial velocity $\frac{\partial u}{\partial t}(0,x)$ of the wave. I'll leave it as an exercise to find out exactly what the formula is, and to see why F and G will be Schwartz if the initial position and velocity are Schwartz.).

(b) Conclude that $u(t,x)$ must be of the form

$$u(t,x) = f(x+t) + g(x-t)$$

for some functions $f : \mathbf{R} \to \mathbf{C}$ and $g : \mathbf{R} \to \mathbf{C}$. (You may assume without justification that the functions F, G constructed in (a) are Schwartz).

Since F is Schwartz, we can write $F = \hat{f}$ for some Schwartz f, and similarly write $G = \hat{g}$. Thus

$$\hat{u}(t,\xi) = \hat{f}(\xi) e^{2\pi i t \xi} + \hat{g}(\xi) e^{-2\pi i t \xi}.$$

Taking inverse Fourier transforms and using the symmetries of the Fourier transform, we obtain

$$u(t,x) = f(x+t) + g(x-t)$$

as desired.

Problem 4. (a) Let $f : \mathbf{R}^2 \to \mathbf{C}$ be a Schwartz function of two variables. Let $g : \mathbf{R} \to \mathbf{C}$ be the function defined by

$$g(x_1) := \int_{\mathbf{R}} f(x_1, x_2)\ dx_2 \text{ for all } x_1 \in \mathbf{R};$$

this function is sometimes called the *orthogonal projection* of f to the x-axis. Give a formula for the one-dimensional Fourier transform $\hat{g}(\xi)$ of g in terms of the two-dimensional Fourier transform $\hat{f}(\xi_1, \xi_2)$ of f. (You may interchange integrals without justification. Your final formula should not have any integrals in it).

For any $\xi \in \mathbf{R}$, we have

$$\hat{g}(\xi) = \int_{\mathbf{R}} g(x_1) e^{-2\pi i x_1 \xi}\ d\xi = \int_{\mathbf{R}} \int_{\mathbf{R}} f(x_1, x_2) e^{-2\pi i x_1 \xi}\ d\xi.$$

Writing $x_1 \xi$ as $(x_1, x_2) \cdot (\xi, 0)$, we thus see that

$$\hat{g}(\xi) = \int_{\mathbf{R}^2} f(x_1, x_2) e^{-2\pi i (x_1, x_2) \cdot (\xi, 0)}\ d\xi = \hat{f}(\xi, 0).$$

(b) Let $f_1 : \mathbf{R} \to \mathbf{C}$ and $f_2 : \mathbf{R} \to \mathbf{C}$ be Schwartz functions of one variable. Define the function $f : \mathbf{R}^2 \to \mathbf{C}$ by

$$f(x_1, x_2) := f_1(x_1) f_2(x_2);$$

this function is sometimes called the *tensor product* of f_1 and f_2. Give a formula for the two-dimensional Fourier transform $\hat{f}(\xi_1, \xi_2)$ of f in terms of the one-dimensional Fourier transforms of f_1 and f_2. (Again, you may interchange integrals without justification. Your final formula should not have any integrals in it).

We have

$$\begin{aligned}\hat{f}(\xi_1, \xi_2) &= \int_{\mathbf{R}^2} f(x_1, x_2) e^{-2\pi i (x_1, x_2) \cdot (\xi_1, \xi_2)}\ dx \\ &= \int_{\mathbf{R}} \int_{\mathbf{R}} f_1(x_1) f_2(x_2) e^{-2\pi i x_1 \xi_1} e^{2\pi i x_2 \xi_2}\ dx_1 dx_2 \\ &= \left(\int_{\mathbf{R}} f_1(x_1) e^{-2\pi i x_1 \xi_1}\ dx_1\right)\left(\int_{\mathbf{R}} f_1(x_2) e^{-2\pi i x_2 \xi_2}\ dx_2\right) \\ &= \hat{f}_1(\xi_1) \hat{f}_2(\xi_2).\end{aligned}$$

Problem 5. (a) Let $a > 0$ be a real number, and let $f : \mathbf{R} \to \mathbf{C}$ be the function $f(x) := e^{-2\pi a|x|}$. Prove that $\hat{f}(\xi) = \frac{a}{\pi(\xi^2+a^2)}$. (This function $\hat{f}$ is sometimes known as the *Cauchy distribution* corresponding to the parameter a, and occurs sometimes in probability theory).

We have

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} e^{-2\pi a|x|} e^{-2\pi i x\xi}\, dx = \int_0^\infty e^{-2\pi x(a+i\xi)}\, dx + \int_{-\infty}^0 e^{-2\pi i x(-a+i\xi)}\, dx$$

$$= \frac{e^{-2\pi x(a+i\xi)}}{-2\pi(a+i\xi)}\Big|_0^\infty + \frac{e^{-2\pi x(-a+i\xi)}}{-2\pi(-a+i\xi)}\Big|_{-\infty}^0.$$

Now observe that as $x \to +\infty$, the function $e^{-2\pi x(a+i\xi)}$ has magnitude $e^{-2\pi a x}$ and hence goes to zero (since $a > 0$). Similarly as $x \to -\infty$, the function $e^{-2\pi x(-a+i\xi)}$ goes to zero. Thus we have

$$\hat{f}(\xi) = \frac{1}{-2\pi(a+i\xi)} - \frac{1}{-2\pi(-a+i\xi)} = -\frac{2a}{2\pi(a+i\xi)(-a+i\xi)} = \frac{a}{\pi(\xi^2+a^2)}$$

as desired.

(b) Using part (a), prove the identities

$$\int_{-\infty}^{\infty} \frac{d\xi}{\xi^2+a^2} = \frac{\pi}{a}$$

and

$$\int_{-\infty}^{\infty} \frac{d\xi}{(\xi^2+a^2)^2} = \frac{\pi}{2a^3}.$$

(You may assume without justification that the Fourier inversion and Plancherel theorems apply not only to Schwartz functions, but more generally to functions of moderate decrease; cf. Section 1.7 of Chapter 5).

From the Fourier inversion formula we have

$$f(0) = \int_{-\infty}^{\infty} \hat{f}(\xi)\, d\xi = \frac{\pi}{a} \int_{-\infty}^{\infty} \frac{d\xi}{\xi^2+a^2};$$

since $f(0) = 1$, the first claim follows. Meanwhile, from Plancherel's theorem we have

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx = \int_{-\infty}^{\infty} |\hat{f}(\xi)|^2 = \frac{\pi^2}{a^2} \int_{-\infty}^{\infty} \frac{dx}{(\xi^2+a^2)^2}.$$

On the other hand,

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx = \int_{-\infty}^{\infty} e^{-4\pi a|x|}\, dx = 2\int_0^\infty e^{-4\pi a x}\, dx = 2\frac{e^{-4\pi i a x}}{-4\pi a}\Big|_0^\infty = \frac{1}{2\pi a}$$

and the second claim follows.

Problem 6. Let $n \geq 1$. For each $t > 0$, let $H_t : \mathbf{R}^n \to \mathbf{C}$ denote the function

$$H_t(x) := \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t}.$$

(This function is sometimes known as the *heat kernel* at time t in n dimensions, and is related to the heat equation). Prove that for every $s, t > 0$ we have the identity

$$H_s * H_t = H_{s+t}.$$

(Hint: There are two ways to proceed. One is direct computation of the convolution. The other proceeds by first computing the Fourier transform of H_t.)

Starting with the fact that the Fourier transform of the Gaussian $e^{-\pi|x|^2}$ is the Gaussian $e^{-\pi|\xi|^2}$, and dilating x by $\sqrt{4\pi t}$, we see that the Fourier transform of H_t is $e^{-4\pi^2|\xi|^2 t}$. Similarly the Fourier transform of H_s is $e^{-4\pi^2|\xi|^2 s}$. Thus the Fourier transform of $H_t * H_s$ is $e^{-4\pi^2|\xi|^2(s+t)}$, which is the Fourier transform of H_{s+t}. Since the Fourier transform is a bijection on Schwartz functions, we obtain $H_t * H_s = H_{s+t}$ as desired.

Problem 7. Let $N \geq 1$ be an integer, and let $f : \mathbf{Z}/N\mathbf{Z} \to \mathbf{C}$ be a function on the cyclic group $\mathbf{Z}/N\mathbf{Z}$. Suppose we take the Fourier transform of f twice, creating a function $\hat{\hat{f}} : \mathbf{Z}/N\mathbf{Z} \to \mathbf{C}$. Find a simple formula relating this function to the original function f. (Your final formula should not have any summations in it. Be warned that the formula may look slightly different from the corresponding formula for the Fourier transform on $\mathbf{R}$, which you worked out as one of your homework problems).

We have

$$\hat{\hat{f}}(x) = \frac{1}{N} \sum_{\xi \in \mathbf{Z}/N\mathbf{Z}} \hat{f}(\xi) e^{-2\pi i x\xi/N}.$$

Comparing this with the Fourier inversion formula

$$f(x) = \sum_{\xi \in \mathbf{Z}/N\mathbf{Z}} \hat{f}(\xi) e^{2\pi i x\xi/N}$$

we see that $\hat{\hat{f}}(x) = \frac{1}{N} f(-x)$. (One can also use $f(N-x)$ instead of $f(-x)$ if wants to keep the variable ranging between 0 and $N-1$ (or between 1 and N).

Problem 8. Let $f : \mathbf{R}^n \to \mathbf{C}$ be a Schwartz function. Show that f is real-valued (i.e. $f(x) \in \mathbf{R}$ for all $x \in \mathbf{R}^n$) if and only if we have $\hat{f}(-\xi) = \overline{\hat{f}(\xi)}$ for all $\xi \in \mathbf{R}^n$. (Hint: f is real-valued if and only if $f(x) = \overline{f(x)}$ for all $x \in \mathbf{R}^n$).

Observe that for any Schwartz f we have

$$\begin{aligned}
\overline{\hat{f}(\xi)} &= \overline{\int_{\mathbf{R}^n} f(x)e^{-2\pi i x\xi}\, d\xi} \\
&= \int_{\mathbf{R}^n} \overline{f(x)}e^{2\pi i x\xi}\, d\xi \\
&= \hat{\overline{f}}(-\xi).
\end{aligned}$$

Thus if f is real, then $f = \overline{f}$, and hence $\hat{f}(-\xi) = \overline{\hat{f}(\xi)}$. Conversely, if $\hat{f}(-\xi) = \overline{\hat{f}(\xi)}$, then $\hat{f}(-\xi) = \hat{\overline{f}}(-\xi)$ for all ξ, which implies that f and $\overline{f}$ have the same Fourier transforms. Since the Fourier transform is a bijection on Schwartz spaces, we have $f = \overline{f}$, thus f is real.

Problem 9. Let N be an odd integer, and let $f : \mathbf{Z}/N\mathbf{Z} \to \mathbf{C}$ be the function $f(x) := e^{2\pi i x^2/N}$. (a) Show that the Fourier transform $\hat{f} : \mathbf{Z}/N\mathbf{Z} \to \mathbf{C}$ of f obeys the identity

$$\hat{f}(2\xi) = e^{-2\pi i \xi^2/N} \hat{f}(0)$$

for all $\xi \in \mathbf{Z}/N\mathbf{Z}$. (Hint: complete the square).

We have

$$\hat{f}(2\xi) = \frac{1}{N} \sum_{x \in \mathbf{Z}/N\mathbf{Z}} e^{2\pi i x^2/N} e^{-2\pi i (2\xi) x/N}.$$

Since $x^2 + (2\xi)x = (x+\xi)^2 - \xi^2$, we thus have

$$\hat{f}(2\xi) = \frac{1}{N} \sum_{x \in \mathbf{Z}/N\mathbf{Z}} e^{2\pi i (x+\xi)^2/N} e^{-2\pi i \xi^2/N}.$$

Making the change of variables $y = x + \xi$ (and noting that if you shift all the elements of $\mathbf{Z}/N\mathbf{Z}$ by ξ, you just get $\mathbf{Z}/N\mathbf{Z}$ again (with each element of $\mathbf{Z}/N\mathbf{Z}$ appearing exactly once) we thus have

$$\hat{f}(2\xi) = \frac{1}{N} \sum_{y \in \mathbf{Z}/N\mathbf{Z}} e^{2\pi i y^2/N} e^{-2\pi i \xi^2/N}.$$

Since $\hat{f}(0) = \frac{1}{N} \sum_{y \in \mathbf{Z}/N\mathbf{Z}} e^{2\pi i y^2/N}$, the claim follows.

(b) Using (a) and Plancherel's theorem, conclude that $|\hat{f}(\xi)| = N^{-1/2}$ for all ξ. (You may use without proof the fact that when N is odd, the map $\xi \mapsto 2\xi$ is a bijection from $\mathbf{Z}/N\mathbf{Z}$ to $\mathbf{Z}/N\mathbf{Z}$). The quantities $\hat{f}(\xi)$ studied here are sometimes called *Gauss sums*, and play an important role in number theory.

From (a) we have $|\hat{f}(2\xi)| = |\hat{f}(0)|$ for all $\xi \in \mathbf{Z}/N\mathbf{Z}$; changing variables $\eta = 2\xi$ we thus see that $|\hat{f}(\eta)| = |\hat{f}(0)|$ for all $\eta \in \mathbf{Z}/N\mathbf{Z}$ (using the above-mentioned bijection property). Thus by Plancherel

$$\frac{1}{N} \sum_{x \in \mathbf{Z}/N\mathbf{Z}} |f(x)|^2 = \sum_{\eta \in \mathbf{Z}/N\mathbf{Z}} |\hat{f}(\eta)|^2 = \sum_{\eta \in \mathbf{Z}/N\mathbf{Z}} |\hat{f}(0)|^2 = N|\hat{f}(0)|^2.$$

On the other hand, we have

$$\frac{1}{N} \sum_{x \in \mathbf{Z}/N\mathbf{Z}} |f(x)|^2 = \frac{1}{N} \sum_{x \in \mathbf{Z}/N\mathbf{Z}} |e^{2\pi i x^2/N}|^2 = \frac{1}{N} \sum_{x \in \mathbf{Z}/N\mathbf{Z}} 1 = 1,$$

and the claim follows.
