Assignment 7 (Due Mar 18). Covers: pages 219-226 of text. Optional reading: pages 226-235.

The questions marked "Optional" are more challenging, and will not count toward your final grade. They will however strengthen both your technical skills and your conceptual understanding of the material.

- Q1. Do Exercise 8 of Chapter 7 of the textbook.

- Q2. In this question we consider solutions to the constant coefficient ordinary differential equation (ODE)

$$a_n \frac{d^n f(x)}{dx^n} + a_{n-1} \frac{d^{n-1} f(x)}{dx^{n-1}} + \ldots + a_1 \frac{df}{dx} + a_0 f = 0$$

  where $a_0, \ldots, a_n$ are complex constants, and $f : \mathbf{R} \to \mathbf{C}$ is a smooth function. We refer to the function f as a *solution* to the above ODE.

- Q2(a). Show that if f and g are solutions to the above ODE, then $f+g$ is also a solution. Also, if c is any complex number and f is a solution to the ODE, then cf is also a solution. (In other words, the space of solutions to the ODE is a vector space).

- Q2(b). Show that if $f(x)$ is a solution to the above ODE, and h is any real number, then the function $g(x)$ defined by $g(x) := f(x-h)$ is also a solution to the above ODE. (In other words, the space of solutions to the ODE is *translation-invariant*).

- Q2(c). (Optional) Suppose f is a solution to the above ODE, and suppose $g : \mathbf{R} \to \mathbf{C}$ is a smooth function such that $g(x)$ is only non-zero on a bounded interval $[a, b]$ (i.e. $g(x) = 0$ whenever $x < a$ or $x > b$; such functions are also called *compactly supported*). Explain why the convolution $f * g$ of f and g is also a solution to the above ODE. (You may interchange derivatives and integrals without justification).

- Q2(d). (Optional) Can you give an informal explanation as to why Q2(abc) are true using the Fourier transform (pretending for now that solutions f to the above ODE have a Fourier transform, even though they may not necessarily be in the Schwartz class)?

- Q3. Let $f : \mathbf{Z}/N\mathbf{Z} \to \mathbf{C}$ be a function on the cyclic group $\mathbf{Z}/N\mathbf{Z}$ (which we think of as the set $\{0, \ldots, N-1\}$ with a modified addition law), and let $\hat f : \mathbf{Z}/N\mathbf{Z} \to \mathbf{C}$ be its Fourier transform, thus

$$\hat f(\xi) = \frac{1}{N} \sum_{x=0}^{N-1} f(x) e^{-2\pi i x \xi/N}.$$

- Q3(a). Let $h \in \mathbf{Z}/N\mathbf{Z}$, and let g be the function $g(x) := f(x-h)$. Derive a formula connecting the Fourier transform of g with the Fourier transform of f (it should resemble Proposition 1.2(i) of Chapter 5, Proposition 2.1(i) of Chapter 6, or Question 4(b) of Assignment 2).

- Q3(b). Let $h \in \mathbf{Z}/N\mathbf{Z}$, and let g be the function $g(x) := f(x)e^{2\pi i x h/N}$. Again, derive a formula connecting the Fourier transform of g with the Fourier transform of f (compare with Proposition 1.2(ii) of Chapter 5 or Proposition 2.1(ii) of Chapter 6).

- Q3(c). (Optional) Let $h \in \mathbf{Z}/N\mathbf{Z}$ be coprime to N (i.e. h and N have no common factor). Show that the map $x \mapsto hx$ is a bijection from $\mathbf{Z}/N\mathbf{Z}$ to $\mathbf{Z}/N\mathbf{Z}$. Conclude in particular that there exists an element $h^{-1} \in \mathbf{Z}/N\mathbf{Z}$ such that $h^{-1}h = hh^{-1} = 1 \mathrm{mod} N$. Now let g be the function $g(x) := f(h^{-1}x)$. Derive a formula connecting the Fourier transform of g with that of f (compare with Proposition 1.2(iii) of Chapter 5 or Proposition 2.1(iii)/(vi) of Chapter 6). What goes wrong when h is not coprime to N?

- Q4. Let $f : \mathbf{Z}/N\mathbf{Z} \to \mathbf{C}$ be a function such that $f(x)$ is non-zero for exactly A different values of x, for some $1 \leq A \leq N$. (Thus, $f(x)$ is zero for the other $N-A$ values of x). Suppose also that $\hat f(\xi)$ is non-zero for exactly B different values of ξ.

- Q4(a). Prove the inequality

$$\sum_{\xi=0}^{N-1} |\hat f(\xi)|^2 \leq B\Big(\max_{\xi=0,\ldots,N-1} |\hat f(\xi)|\Big)^2.$$

- Q4(b). Prove the inequality

$$\max_{\xi=0,\dots,N-1} |\hat{f}(\xi)| \leq \frac{1}{N}\sum_{x=0}^{N-1} |f(x)|.$$

- Q4(c). Prove the inequality

$$\sum_{x=0}^{N-1} |f(x)| \leq A^{1/2}(\sum_{x=0}^{N-1} |f(x)|^2)^{1/2}.$$

  (Hint: Use the Cauchy-Schwarz inequality).

- Q4(d). Using Q4(abc) and Plancherel's theorem, prove the inequality

$$AB \geq N.$$

  This is known as the *uncertainty principle for* $\mathbf{Z}/N\mathbf{Z}$; informally, it means that a function f and its Fourier transform $\hat{f}$ cannot simultaneously have a "narrow support".

- Q5. If $f : \mathbf{Z}/N\mathbf{Z} \to \mathbf{C}$ and $g : \mathbf{Z}/N\mathbf{Z} \to \mathbf{C}$ are two functions, define the convolution $f * g : \mathbf{Z}/N\mathbf{Z} \to \mathbf{C}$ by the formula

$$f * g(x) := \frac{1}{N}\sum_{y=0}^{N-1} f(y)g(x-y).$$

- Q5(a). Prove that $\widehat{f * g}(\xi) = \hat{f}(\xi)\hat{g}(\xi)$.

- Q5(b). What is the relationship between $\widehat{fg}(\xi)$ and $\widehat{f} * \widehat{g}(\xi)$? (note: it is slightly different from the situation in Q5(a)).

- Q5(c). Can you find a function $\delta : \mathbf{Z}/N\mathbf{Z} \to \mathbf{C}$ such that $f * \delta = \delta * f = f$ for all functions $f : \mathbf{Z}/N\mathbf{Z} \to \mathbf{C}$? If so, what is this function δ? What is the Fourier transform of this function δ? (You may wish to answer the second question first).

- Q6. Let $f : \mathbf{Z}/N\mathbf{Z} \to \mathbf{C}$ and $g : \mathbf{Z}/N\mathbf{Z} \to \mathbf{C}$ be functions. Prove *Parseval's identity*

$$\frac{1}{N}\sum_{x=0}^{N-1} f(x)\overline{g(x)} = \sum_{\xi=0}^{N-1} \hat{f}(\xi)\overline{\hat{g}(\xi)}.$$

- Q7. Let N be a positive integer, and suppose N factors as $N = N_1 N_2$ for some smaller integers N_1, N_2. Let $f : \mathbf{Z}/N\mathbf{Z} \to \mathbf{C}$ be the function defined by setting $f(x) = 1$ when x is a multiple of N_1, and $f(x) = 0$ otherwise.

- Q7(a). Show that $\hat{f}(\xi) = 1/N_1$ when ξ is a multiple of N_2, and $\hat{f}(\xi) = 0$ otherwise.

- Q7(b). For any function $g : \mathbf{Z}/N\mathbf{Z} \to \mathbf{C}$, prove the *Poisson summation formula for finite groups*:

$$\frac{1}{N_1}\sum_{x \in \mathbf{Z}/N\mathbf{Z} : x \text{ is a multiple of } N_1} g(x) = \sum_{\xi \in \mathbf{Z}/N\mathbf{Z} : \xi \text{ is a multiple of } N_2} \hat{g}(\xi).$$

  (Optional) Can you find some informal argument that links this formula to the standard Poisson summation formula?