Assignment 5 (Due Mar 4). Covers: pages 140-145, 175-184 of text.

The questions marked "Optional" are more challenging, and will not count toward your final grade. They will however strengthen both your technical skills and your conceptual understanding of the material.

- Q1. Do Exercise 1 of Chapter 6 in the textbook.
- Q2 (a). Let $f(x,y)$ be a Schwartz function of two variables, and let a,b,c,d be real numbers such that $ad-bc\neq 0$. Let g be the function $g(x,y) := f(ax+by, cx+dy)$. Find a formula relating the Fourier transform of g to the Fourier transform of f.
- Q2 (b). What goes wrong if $ad-bc=0$?
- Q2 (c) (Optional: only attempt if you have taken Math 115A or equivalent): Can you generalize the above statements to functions of n variables, with a change of variables given by an $n\times n$ matrix?
- Q3. Let $f(x)$ be a Schwartz function of one variable such that $f*f=f$. Show that f must in fact be zero. (Optional) Is the same statement true for functions of several variables?
- Q4 (a). Compute all the derivatives of the Gaussian $e^{-\pi\xi^2}$ at the point $\xi=0$; in other words, compute $\frac{d^n}{d\xi^n}e^{-\pi\xi^2}|_{\xi=0}$ for $n=1,2,\ldots$. (Hint: Expand the Gaussian as a Taylor series).
- Q4 (b). Compute the integrals $\int_{-\infty}^{\infty}(-2\pi i x)^n e^{-\pi x^2}\,dx$ for $n=1,2,3,\ldots$. (Hint: Use part (a), and the properties of the Fourier transform).
- Q5. Let $f(x)$ and $g(x)$ be Schwartz functions of one variable. Show that $\widehat{fg}=\widehat{f}*\widehat{g}$ (note that this is similar to, but not exactly the same, as Proposition 1.11(iii) of Chapter 5). (Optional) Is the same statement true for functions of several variables?
- Q6. This question concerns the *Hermite operator* $H := -\frac{d^2}{dx^2}+4\pi^2x^2$; thus if $f(x)$ is an Schwartz function of one variable, we let $Hf(x)$ denote the function

$$Hf(x) := -\frac{d^2 f}{dx^2}+4\pi^2x^2 f.$$

If f is a Schwartz function and λ is a real number, we say that f is an *eigenfunction of the Hermite operator with eigenvalue* λ if we have $Hf = \lambda f$.

- Q6(a) Show that the Gaussian $e^{-\pi x^2}$ is an eigenfunction of the Hermite operator. What is the eigenvalue?

- Q6(b) Show that if f is an eigenfunction of the Hermite operator, then the Fourier transform $\hat{f}$ is also an eigenfunction of the Hermite operator, and with the same eigenvalue. (Hint: take the Fourier transform of both sides of the eigenfunction equation $Hf = \lambda f$).

- Q6(c) (Optional) Show that if f is an eigenfunction of the Hermite operator, and g is the function $g := \frac{df}{dx} - 2\pi x f$, then g is also an eigenfunction of the Hermite operator, but with a different eigenvalue (what is the relationship between the two eigenvalues?)

- Q7 (a). Let $f(x)$ denote the function of one variable such that $f(x) = 1$ when $2 < x < 3$, and $f(x) = 0$ otherwise; this type of function is sometimes called a *step function*. Compute $f * f$ and $f * f * f$, and sketch all three functions f, $f * f$, and $f * f * f$.

- Q7 (b). Based on your answer to Q7(a), describe in words what you think happens to higher convolutions $f * f * f * f$, $f * f * f * f * f$, etc. (Sketch graphs, or describe what the function might look like in words. You do not have to supply rigorous proofs of your assertions for this question).

- Q8. Let $f(x, y)$ be the function of two variables such that $f(x, y) = 1$ when (x, y) lives in the unit square $\{(x, y) : 0 < x < 1 \text{ and } 0 < y < 1\}$, and $f(x, y) = 0$ otherwise. Compute $f * f$, and compute its graph. (This is a function of two variables, so the graph will be some sort of surface).

- Q9. Let f and g be Schwartz functions, and let h be the identity function $h(x) = x$.

- Q9(a). Prove the identity $h \times (f * g) = (h \times f) * g + f * (h \times g)$ by using the definition of convolution and the properties of integrals. (We use

$h \times f$ to denote the pointwise product of h and f, thus for instance if $f(x)$ was the Gaussian $f(x) = e^{-\pi x^2}$, then $h \times f$ would be the function $(h \times f)(x) = xe^{-\pi x^2}$. Sometimes this product $h \times f$ is abbreviated as hf).

- Q9(b). Prove the same identity $h \times (f * g) = (h \times f) * g + f * (h \times g)$ by using the properties of Fourier transform (notably Proposition 1.2(v), Corollary 1.10, and Proposition 1.11(iii)).