Assignment 2 (Due January 29). Covers: pages 42-51 of text.

The questions marked "Optional" are more challenging, and will not count toward your final grade. They will however strengthen both your technical skills and your conceptual understanding of the material.

- Q1. Do Exercise 10 of Chapter 2 in the textbook.
- Q2 (Optional). Do Exercise 12 of Chapter 2 in the textbook.
- Q3. Let $N > 0$ be a positive integer, and let $D_N(x)$ be the Dirichlet kernel (defined on page 37 of the textbook).
- Q3(a). What are the Fourier coefficients of D_N?
- Q3(b). Show that $D_N * D_N(x) = D_N(x)$ for all $x \in \mathbf{R}$.
- Q4. Let f be a continuous 2π-periodic function, and let h be a real number. Let f_h be the function defined by $f_h(x) := f(x - h)$.
- Q4 (a). Show that f_h is also a continuous 2π-periodic function. Geometrically, what is the connection between the graph of f and the graph of f_h?
- Q4 (b). Show that $\widehat{f_h}(n) = e^{-inh}\hat{f}(n)$ for all $n \in \mathbf{Z}$.
- Q4 (c). Let g be another continuous 2π-periodic function, and define the function g_h by $g_h(x) := g(x - h)$. Show that $f * g_h(x) = f_h * g(x)$ for all x. (Hint: either use Q4(b), or use a change of variable).
- Q4 (d). Let $(f * g)_h$ be the function defined by $(f * g)_h(x) := (f * g)(x - h)$. Show that $(f * g)_h(x) = f * g_h(x) = f_h * g(x)$ for all x. (You may use Q4(c), of course).
- Q5. Let f be a continuous 2π-periodic function, and let h be a real number. Let $\Delta_h f$ be the function defined by $\Delta_h f(x) := \frac{f(x+h)-f(x)}{h}$. (Note: the symbol Δ_h is being used here not to denote a function or a number - $\Delta_h f$ is not "Δ_h times f" - but instead denotes an *operator* - something which takes a function f as input and returns another function $\Delta_h f$ as output. If you like, an operator is a function whose domain and range themselves consist of functions. Another example of

an operator is $\frac{d}{dx}$ - this is neither a function nor a number, but is an operator that takes a function f as input and returns a new function $\frac{d}{dx}f$ as output.)

- Q5(a). Show that $\Delta_h f$ is also a continuous 2π-periodic function. If f is continuously differentiable, what happens to $\Delta_h f$ as $h \to 0$?

- Q5(b). Show that $\widehat{\Delta_h f}(n) = \frac{e^{inh}-1}{h}\hat{f}(n)$ for all $n \in \mathbf{Z}$. (Hint: you can either compute directly, or use the results of Q4). How does this relate to the identity in the middle of page 43 of the textbook? (Note that you can evaluate the limiting behavior of $\frac{e^{inh}-1}{h}$ as $h \to 0$ using L'hôpital's rule.)

- Q5(c). Let g be another continuous 2π-periodic function. Show that

$$(\Delta_h f) * g(x) = f * (\Delta_h g)(x) = (\Delta_h(f * g))(x)$$

  for all x. (Caution: watch the parentheses carefully, and make sure you know what quantities are functions, what quantities are operators, and what quantities are numbers). (Hint: again, you may use the results of Q4).

- Q6(a). Let f and g be continuously differentiable 2π-periodic functions. Show that

$$f' * g(x) = f * g'(x)$$

  for all x. (Hint: either integrate by parts, or use Q5). Here of course f' and g' are the derivatives of f and g respectively.

- Q6(b). With the assumptions in Q6(a), it is also true that $f * g$ is continuously differentiable, and

$$(f * g)'(x) = f' * g(x) = f * g'(x).$$

  Give at least one informal explanation why this identity should be true. Feel free to hand-wave your way around operations such as taking limits, and don't be too concerned with which functions are known to be differentiable, continuous, etc. (Hint: There are several ways to do this. Some possibilities to try: (i) use Q5(c). (ii) use the identity in the middle of page 43 of the textbook. (iii) differentiate under the integral sign. (iv) use first principles.) Note: to prove this statement rigorously is a little tricky, and may require one to use concepts such as uniform continuity. As an (optional) challenge, you can try giving a rigorous proof instead of a hand-waving one, but I would only recommend this if you have taken a course such as 131B.

- Q6(c). (Optional) Let f and g be 2π-periodic functions. Based on the result in Q6(b), how smooth would you expect $f * g$ to be if you knew that f was continuous and g was continuously differentiable (i.e. would you expect $f * g$ to be continuous, continuously differentiable, or twice continuously differentiable?). More generally, if f is k times continuously differentiable, and g is l times continuously differentiable (for some integers $k, l \geq 0$), how smooth would you expect $f * g$ to be? Give some informal justification for your statements.

- Q7. For each positive integer n, let $K_n(x)$ be the 2π-periodic function defined by $K_n(x) := 2n(1 - \frac{n|x|}{\pi})$ when $|x| \leq \pi/n$, by $K_n(x) := 0$ when $\pi/n < |x| \leq \pi$, and then extended to be 2π-periodic for all other values of x. Sketch K_n for the first few values of n, and show that $\{K_n(x)\}_{n=1}^{\infty}$ is a family of good kernels (as in page 48 of the textbook).

- Q8. Let $\{K_n(x)\}_{n=1}^{\infty}$ be a family of good kernels. Show that for every integer k, we have $\lim_{n\to\infty} \widehat{K_n}(k) = 1$. (Hint: the quickest proof proceeds via Theorem 4.1).

- Q9. Let $\{J_n(x)\}_{n=1}^{\infty}$ and $\{K_n(x)\}_{n=1}^{\infty}$ be two families of good kernels, and let $\{L_n(x)\}_{n=1}^{\infty}$ be the family of functions defined as $L_n := J_n * K_n$. Thus each of the L_n is a continuous 2π-periodic function.

- Q9(a). Show that $\frac{1}{2\pi} \int_{-\pi}^{\pi} L_n(x)\, dx = 1$ for all n. (Hint: Have a look at Proposition 3.1(vi) and its proof).

- Q9(b). Show that there exists a constant M such that $\frac{1}{2\pi} \int_{-\pi}^{\pi} |L_n(x)|\, dx \leq M$ for all n. (You may use without proof the *triangle inequality for integrals*, which asserts that $|\int_a^b f(x)\, dx| \leq \int_a^b |f(x)|\, dx$ for all Riemann integrable functions f on an interval $[a, b]$.)

- Q9(c) (Optional). Show that for every $\delta > 0$, we have $\int_{\delta \leq |x| \leq \pi} |L_n(x)|\, dx \to 0$ as $n \to \infty$. Conclude that convolving two families of good kernels together yields another family of good kernels.

- Q10 (Optional). The purpose of this exercise is to explain ourselves why we must use *approximations* to the identity rather than *exact* identities when it comes to convolutions. Show that there does not exist a continuous 2π-periodic function δ such that $f * \delta = f$ for all continuous 2π-periodic functions f. (Hint: Assume for contradiction that such a function δ exists. Let $\{K_n\}_{n=1}^{\infty}$ be some family of good kernels (e.g. the ones in Q7), and consider what happens to $K_n * \delta$ as $n \to \infty$). Remark: Later on in this course we will discuss the *Dirac delta function* δ, which is an exact identity for convolution in the sense that $f * \delta = f$ for all continuous 2π-periodic f. But the Dirac delta is not a continuous function, in fact strictly speaking it is not even a function; it turns out to belong in the more general class of objects known as *measures*, which in turn are contained in an even more general class of objects known as *distributions*. More on this later in the course.