Math 132 - Week 8
Textbook sections: 6.1-6.3
Topics covered:

- Residues; the residue theorem
- Trig and indefinite integrals
- Indefinite integrals

Integration around a singularity

- Let $f(z)$ be a function with an isolated singularity at z_0, and let γ be a simple closed anti-clockwise contour around z_0 which contains no other singularities. In the past we have used the Cauchy integral formula and its variants to compute $\oint_\gamma f(z)\ dz$; however this formula is not always applicable. For instance, we cannot work out

$$\oint_{|z|=1} e^{1/z} dz$$

using the Cauchy theorems, because the singularity at 0 does not seem to be of the form $\frac{f(z)}{(z-z_0)^k}$.

- However, we can work out the integral using Laurent series. We expand

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \dots$$

integrating this term by term, we get

$$\oint_{|z|=1} e^{1/z} dz = \oint_{|z|=1} dz + \oint_{|z|=1} \frac{dz}{z}$$

$$+ \oint_{|z|=1} \frac{dz}{2!z^2} + \oint_{|z|=1} \frac{dz}{3!z^3} + \dots$$

Despite looking complicated, most of these integrals are zero. Recall that whenever f has an anti-derivative on a closed contour γ, then

$\int_\gamma f(z)\ dz = 0$. With the exception of $\frac{1}{z}$, all the integrands have anti-derivatives (e.g. $\frac{1}{2!z^2}$ has an anti-derivative of $-\frac{1}{2z}$), so they all vanish. Thus we have

$$\oint_{|z|=1} e^{1/z} dz = \oint_{|z|=1} \frac{dz}{z} = 2\pi i$$

by Cauchy's integral formula.

- More generally, if $f(z)$ has a Laurent expansion of

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

  and γ is a simple closed anti-clockwise contour which contains z_0 and no other singularities of f, then

$$\int_\gamma f(z)\ dz = \int_\gamma \frac{a_{-1}}{z - z_0}\ dz = 2\pi i a_{-1}$$

  because all the other terms in the Laurent expansion have anti-derivatives.

- Because the a_{-1} co-efficient is the only thing left after integration, we call it the *residue* of f at z_0, and denote it $Res(f; z_0)$. We can thus rephrase the above result as

$$\int_\gamma f(z)\ dz = 2\pi i Res(f; z_0).$$

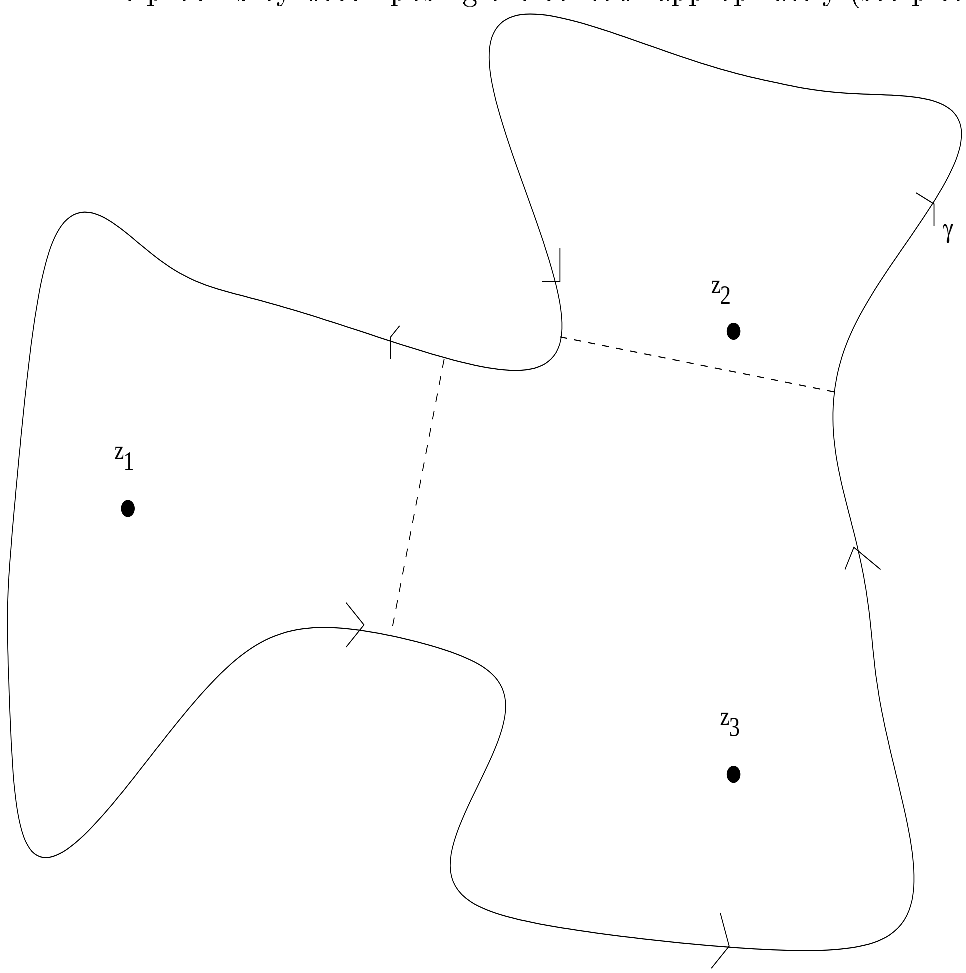
  For instance, we have $Res(e^{1/z}, 0) = 1$.

Integration around multiple singularities

- Now suppose that γ is a simple closed anti-clockwise contour which encloses more than one singularity of f; let's say it encloses n singularities $z_1, z_2, \ldots, z_n$. Then we have

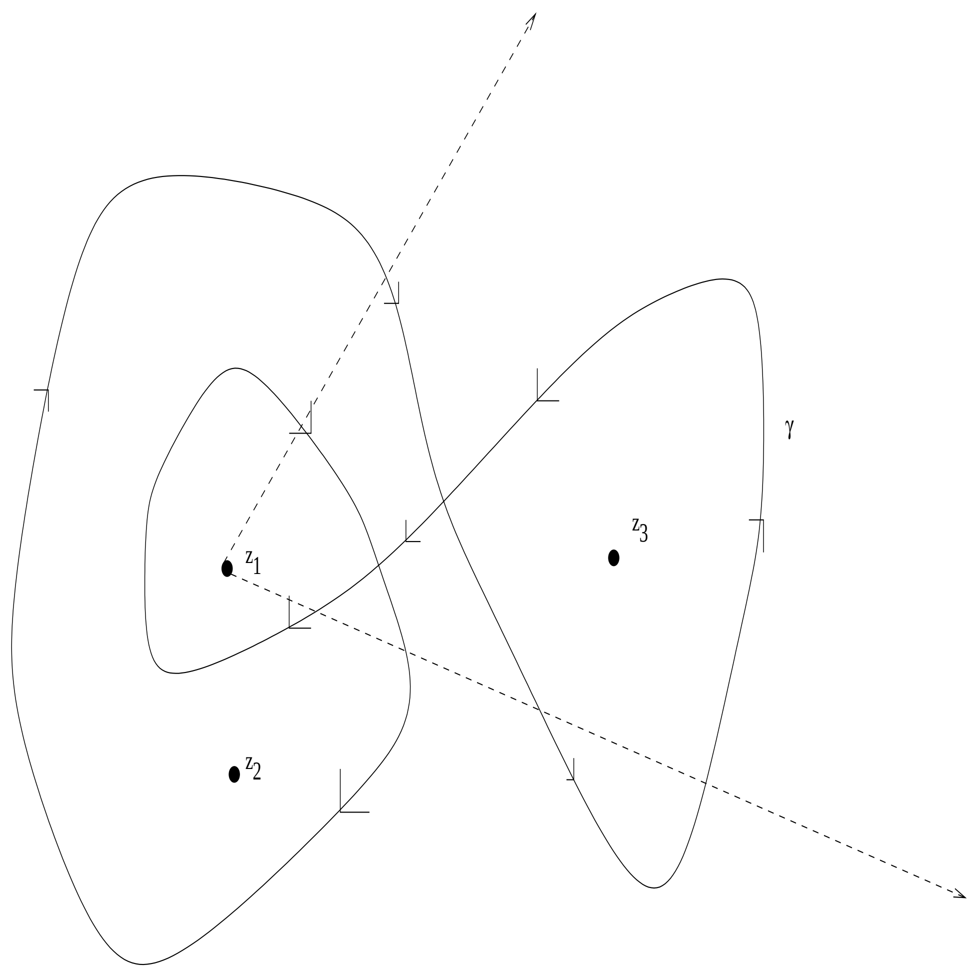
$$\int_\gamma f(z)\ dz = 2\pi i Res(f; z_1) + 2\pi i Res(f; z_2)$$
$$+ \ldots + 2\pi i Res(f; z_n).$$

- The proof is by decomposing the contour appropriately (see picture).



Non-simple or clockwise contours

- Now suppose that γ is a more complicated closed contour, which encloses one or more singularities $z_1, z_2, \ldots, z_n$.



- In the above situation, we have

$$\int_\gamma f(z)\ dz = 2\pi i(-2Res(f;z_1) - Res(f;z_2) + Res(f;z_3))$$

  by decomposing the contour into pieces.

- More generally, we need to define the notion of a *winding number*. Informally, the winding number of a contour γ around a point z_0 is the number of times γ winds anti-clockwise around z_0 (this number can be negative if γ goes clockwise). A more precise definition is as follows. Take any path starting at z_0 and heading out toward infinity. As one follows this path, take the number of times γ hits this path from the right, minus the number of times γ hits this path from the left. This is the winding number of γ around z_0; it turns out that no matter what path you choose, the winding number is always the same.

- The winding number of γ around z_0 is sometimes denoted $Wind(\gamma; z_0)$. The most general result about integration on closed contours is

- **Residue Theorem.** Let D be a simply connected domain, and let f be a function on D which is analytic on D except at a finite number of isolated singularities $z_1, z_2, \ldots, z_n$. Then we have

$$\int_\gamma f(z)\ dz = 2\pi i \sum_{j=1}^n Res(f; z_j) Wind(\gamma; z_j)$$

  for all closed contours γ in D.

- The proof of this theorem is beyond the scope of this course, requiring a certain amount of topology, but by trying examples such as the one above one can be convinced of its validity.

- With this theorem, one can compute the integral of f on closed contours extremely quickly, provided that one knows all the residues of f.

Computation of residues

- In light of the residue theorem, it is clearly of interest to find a way to compute residues quickly. The method to compute residues depends on the nature of the singularity.

- Let's first suppose that f has a removable singularity at z_0. Then the Laurent expansion of f around z_0 has no negative powers of $(z - z_0)$. In particular, there is no $\frac{1}{z-z_0}$ term, and so the residue at a removable singularity is zero.

- Now suppose f has a simple pole at z_0, so the Laurent expansion looks like

$$f(z) = \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + \ldots$$

The residue is a_{-1}. To find it, we multiply the above expansion by $z - z_0$:

$$f(z)(z - z_0) = a_{-1} + a_0(z - z_0) + a_1(z - z_0)^2 + \dots$$

taking the limit as $z \to z_0$, we get

$$\lim_{z \to z_0} f(z)(z - z_0) = a_{-1}.$$

Thus we can compute the residue at simple poles by

$$Res(f; z_0) = \lim_{z \to z_0} f(z)(z - z_0).$$

- For instance, we can compute the residue of $\frac{1}{\sin(z)}$ at the simple pole 0 by
$$Res(\frac{1}{\sin(z)}; 0) = \lim_{z \to 0} \frac{z}{\sin(z)} = \lim_{z \to 0} \frac{1}{\cos(z)} = 1.$$

- Now, suppose f has a double pole at z_0, so the Laurent expansion looks like
$$f(z) = \frac{a_{-2}}{(z - z_0)^2} + \frac{a_{-1}}{(z - z_0)} + a_0 + \dots$$
Now the previous trick to extract the residue a_{-1} won't work, because $f(z)(z - z_0)$ does not converge as $z \to z_0$. However, one can still recover the residue with a bit more work. We multiply $f(z)$ by $(z - z_0)^2$:
$$f(z)(z - z_0)^2 = a_{-2} + a_{-1}(z - z_0) + a_0(z - z_0)^2 + \dots$$
To get rid of a_{-2}, we differentiate both sides with respect to z:
$$\frac{d}{dz}(f(z)(z - z_0)^2) = a_{-1} + 2a_0(z - z_0) + \dots$$
now one can take limits to obtain a_{-1}:
$$Res(f(z); z_0) = \lim_{z \to z_0} \frac{d}{dz}(f(z)(z - z_0)^2)$$

- One can do the same type of trick for higher order poles; if f has a pole of order k at z_0 then

$$Res(f(z); z_0) = \lim_{z \to z_0} \frac{1}{(k-1)!} \frac{d^k}{dz^k}(f(z)(z-z_0)^k).$$

However, this formula is quite cumbersome to use. Usually it is better to seek a different way to compute a_{-1} (e.g. by computing the entire Laurent series).

- When f has an essential singularity, there is no quick formula to compute the residue. Often the only way is to first compute the entire Laurent series and then extract the $1/(z-z_0)$ co-efficient.

- Now, we'll show how residue calculus can also be used to compute *real* definite integrals as well as complex contour integrals. For instance, we'll be able to compute such integrals as

$$\int_0^\pi \frac{d\theta}{2+\cos(\theta)}$$

$$\int_0^\infty \frac{dx}{x^4+1}$$

$$\int_0^\infty \frac{\cos(x)\ dx}{x^4+1}$$

$$p.v. \int_{-\infty}^\infty \frac{e^{ix}\ dx}{x^3+x}$$

$$p.v. \int_0^\infty \frac{dx}{\sqrt{x}(x+4)}$$

exactly; all of these integrals are extremely difficult to solve by conventional techniques! Thus contour integration is a powerful new tool to evaluate real integrals, although it is not a magic wand and cannot handle every single integral in existence.

- I should warn you in advance that this is a rather difficult section, but also one of the most important in the course. You should go through the notes carefully after the lecture.

Definite trigonometric integrals

- The first type of integral we will consider is a trigonometric integral on an interval of length 2π; that is, an integral of the form

$$\int_{\alpha}^{\alpha+2\pi} f(\cos\theta, \sin\theta)\ d\theta$$

where $f(\cos\theta, \sin\theta)$ is some combination of $\cos(\theta)$ and $\sin(\theta)$.

- To illustrate the method, we take as an example

$$\int_0^{2\pi} \frac{d\theta}{2+\cos(\theta)}\ d\theta.$$

- To compute this integral using residue calculus techniques, the first step is to replace the sines and cosines with complex exponentials using the formulae (see Week 3)

$$\cos(\theta) = \frac{e^{i\theta}+e^{-i\theta}}{2}, \quad \sin(\theta) = \frac{e^{i\theta}-e^{-i\theta}}{2i}.$$

The above integral then becomes

$$\int_0^{2\pi} \frac{d\theta}{2+\frac{1}{2}e^{i\theta}+\frac{1}{2}e^{-i\theta}}.$$

- Now we make the complex substitution $z = e^{i\theta}$, so $dz = ie^{i\theta}d\theta$. Since θ ranges from 0 to 2π, the variable z clearly traverses the curve $z = e^{i\theta}, 0 \leq \theta \leq 2\pi$, which is the unit circle $|z| = 1$ traversed once anticlockwise. The integral then becomes

$$\oint_{|z|=1} \frac{dz/(ie^{i\theta})}{2+\frac{1}{2}e^{i\theta}+\frac{1}{2}e^{-i\theta}}$$

which we can write in terms of $z = e^{i\theta}$ as

$$\oint_{|z|=1} \frac{dz/(iz)}{2+\frac{1}{2}z+\frac{1}{2}\frac{1}{z}}.$$

This simplifies to

$$\oint_{|z|=1} \frac{2dz}{4iz + iz^2 + i},$$

or

$$\frac{2}{i} \oint_{|z|=1} \frac{dz}{z^2 + 4z + 1}.$$

- To summarize so far, we have converted a definite integral of a real trigonometric expression into a contour integral of a rational expression on a closed contour. Now we use residue calculus to compute this integral.

- The first step is to factorize the denominator

$$z^2 + 4z + 1 = (z + 2 - \sqrt{3})(z + 2 + \sqrt{3})$$

so our integral becomes

$$\frac{2}{i} \oint_{|z|=1} f(z)\ dz$$

where

$$f(z) = \frac{1}{(z + 2 - \sqrt{3})(z + 2 + \sqrt{3})}.$$

- At this point we could use partial fractions if desired, but we'll use residue calculus instead. The function $f(z)$ has singularities at $-2 + \sqrt{3}$ and $-2 - \sqrt{3}$, but the contour $|z| = 1$ only goes around the first singularity. Since our contour is simple and anti-clockwise, we can compute the integral using the residue theorem as

$$\frac{2}{i} 2\pi i Res(f; -2 + \sqrt{3}).$$

- To compute the residue we have to see what kind of singularity f has at $-2 + \sqrt{3}$. The denominator of f clearly has a simple zero at $-2 + \sqrt{3}$ (one can only divide out one power of $z + 2 - \sqrt{3}$ before becoming discontinuous), so f itself has a simple pole. Thus we can use the formula

$$Res(f; z_0) = \lim_{z \to z_0} f(z)(z - z_0)$$

so that our integral is equal to

$$\frac{2}{i}2\pi i \lim_{z\to -2+\sqrt{3}} f(z)(z+2-\sqrt{3})$$

$$= 4\pi \lim_{z\to -2+\sqrt{3}} \frac{1}{z+2+\sqrt{3}}$$

$$= \frac{2\pi}{\sqrt{3}}.$$

Thus we have computed

$$\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \frac{2\pi}{\sqrt{3}}.$$

- Note that even though complex numbers made an appearance during the middle of our computation, they disappeared at the end. This has to happen, of course, since if one integrates a real function on a real interval one can only get a real number! If our computation of this integral ended up with an i somewhere then we know that we've made a mistake.

- This technique only works when you integrate on an interval of length 2π, or of some multiple of 2π. For instance, it cannot directly handle the integral
$$\int_0^{\pi} \frac{d\theta}{2+\cos\theta}$$
because when one makes the change of variables $z = e^{i\theta}$ one only ends up with a semi-circle instead of a circle - not a closed contour! (One can try the change of variables $z = e^{2i\theta}$ instead, but that has its own problems because one starts encountering expressions like $p.v.z^{1/2}$, which have non-isolated singularities).

- However, sometimes one can handle these expressions by a symmetry argument. For instance, the function $\cos\theta$ is symmetric around $\theta = \pi$, so the above integral is equal to half of the integral from 0 to 2π. Thus
$$\int_0^{\pi} \frac{d\theta}{2+\cos\theta} = \frac{\pi}{\sqrt{3}}.$$

- Challenge: see if you can compute these integrals without using complex methods (and without using Maple).

Integrating rational functions on the real line

- Now we consider a quite different type of integral, namely the integral of a rational function $P(x)/Q(x)$ on the entire real line:

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\, dx.$$

An example is

$$\int_{-\infty}^{\infty} \frac{dx}{x^4+1}\, dx.$$

(Note that this integral converges: for $x > 1$ or $x < 1$ one can use the comparison test using $1/x^4$, and for $-1 \leq x \leq 1$ one can use the comparison test using 1).

- To handle the indefinite integral, we write it as a limit of definite integrals:

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{dx}{x^4+1}.$$

We can write this as

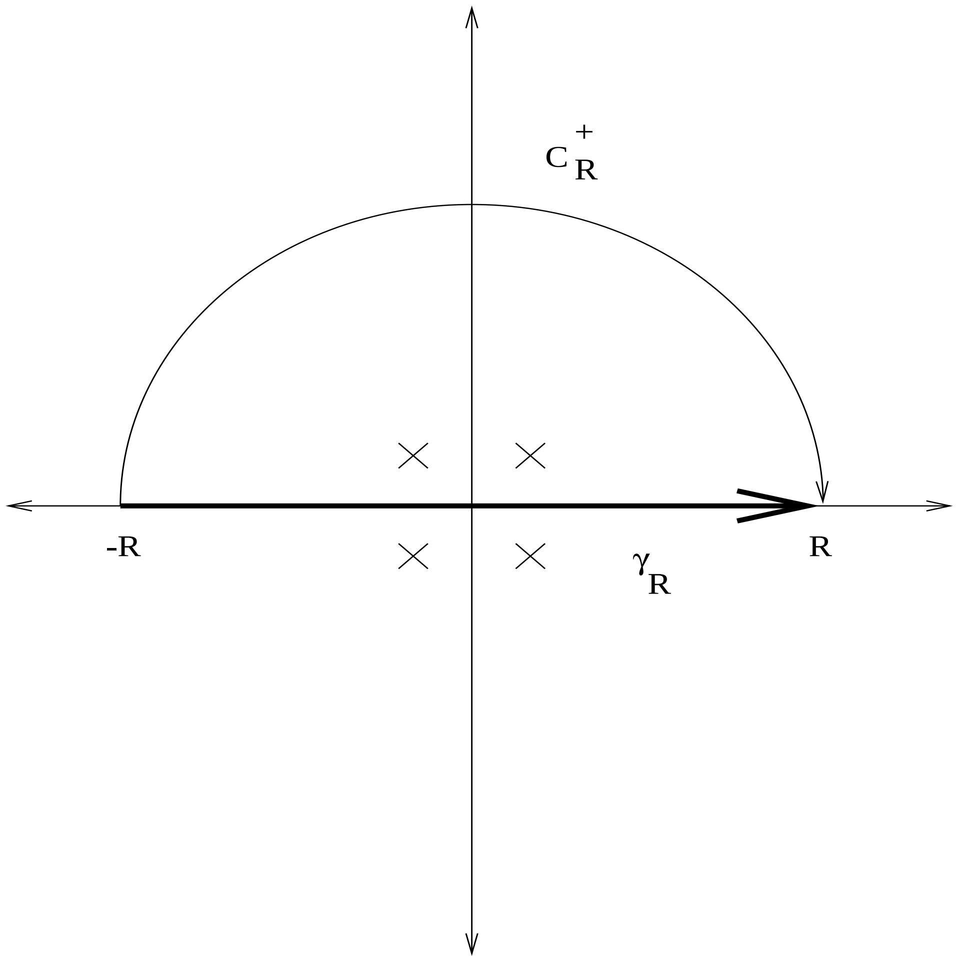
$$\lim_{R\to\infty} \int_{\gamma_R} f(z)\, dz$$

where γ_R is the straight line from $-R$ to R and $f(z) = \frac{1}{z^4+1}$.

- The function f is somewhat large near the origin (for instance, $f(0) = 1$) but decreases quite quickly as one moves away from the origin. (For instance, $f(10)$, $f(-10)$, $f(10i)$, and $f(-10i)$ are all roughly equal to 0.0001.).

- We now use a very versatile technique known as *shifting the contour*; the idea is to try to push the integral away from the origin, where it is large, out toward infinity, where the integral is small. In our particular case we shall shift the contour onto a large semi-circle; this is known as the method of expanding semi-circular contours.

- Let C_R^+ denote the semi-circle $z = Re^{i(\pi - t)} : 0 \leq t \leq \pi$ from $-R$ to R; this contour has the same endpoints as γ_R.



- If f had no singularities, then we could use Cauchy-Goursat to shift the contour and conclude that

$$\int_{\gamma_R} f(z)\ dz = \int_{C_R^+} f(z)\ dz.$$

However, there are some singularities, and so there are some correction terms coming from residues.

- Let's first work out where the singularities of $f(z)$. They occur when

$$z^4 + 1 = 0$$
$$z^4 = -1$$
$$z^4 = e^{\pi i + 2k\pi i}$$
$$z = e^{(\pi i + 2k\pi i)/4}$$
$$z = e^{\pi i/4}, e^{3\pi i/4}, e^{5\pi i/4}, e^{7\pi i/4}.$$

  This are the four crosses on the above diagram (we can assume that R is so large that the semi-circle contains the upper two singularities, since we're eventually going to let $R \to \infty$).

- Now we use residue calculus to compute the integral

$$\int_{\gamma_R + -C_R^+} f(z)dz.$$

  This contour winds around the singularities $e^{\pi i/4}$ and $e^{3\pi i/4}$ once anti-clockwise, but does not wind around the other two singularities at all. Thus

$$\int_{\gamma_R + -C_R^+} f(z)\ dz = 2\pi i Res(f; e^{\pi i/4}) + 2\pi i Res(f; e^{3\pi i/4})$$

- To compute these residues, we need to first find out the nature of the singularities at $e^{\pi i/4}$ and $e^{3\pi i/4}$.

- The function $z^4 + 1$ is zero at $e^{\pi i/4}$ and $e^{3\pi i/4}$, but its derivative $4z^3$ is non-zero at both of these places. Thus $z^4 + 1$ has a simple zero at these points, so $f(z) = 1/(z^4 + 1)$ has a simple pole. We can thus use the simple pole formula for residues:

$$Res(f; e^{\pi i/4}) = \lim_{z \to e^{\pi i/4}} \frac{z - e^{\pi i/4}}{z^4 + 1}.$$

  Both the numerator and denominator tend to zero, so we may use L'hopital's rule:

$$Res(f; e^{\pi i/4}) = \lim_{z \to e^{\pi i/4}} \frac{1}{4z^3}$$

$$= \frac{1}{4}e^{-3\pi i/4} = -\frac{1}{4\sqrt{2}} - \frac{i}{4\sqrt{2}}.$$

A similar computation gives

$$Res(f; e^{\pi i/4}) = \lim_{z \to e^{3\pi i/4}} \frac{1}{4z^3}$$

$$= \frac{1}{4}e^{-9\pi i/4} = \frac{1}{4\sqrt{2}} - \frac{i}{4\sqrt{2}}.$$

- Combining all these calculations, we get

$$\int_{\gamma_R + -C_R^+} f(z)\ dz = 2\pi i(-\frac{i}{2\sqrt{2}} = \frac{\pi}{\sqrt{2}}.$$

- We can thus compare the γ_R integral and C_R^+ integral as

$$\int_{\gamma_R} f(z)\ dz = \frac{\pi}{\sqrt{2}} + \int_{C_R^+} f(z)\ dz.$$

- Taking limits, we get

$$\int_{-\infty}^{\infty} \frac{dx}{x^4+1} = \frac{\pi}{\sqrt{2}} + \lim_{R\to\infty} \int_{C_R^+} \frac{dz}{z^4+1}.$$

- To summarize so far, we've managed to shift the contour from the real line to a big semicircle C_R^+, while picking up some residues totaling $\frac{\pi}{\sqrt{2}}$. The next step is to show the integral on this semi-circle actually goes to zero as $R \to \infty$. That's because z is very large on C_R^+, and $\frac{1}{z^4+1}$ is very small.

- More precisely, when z is on C_R^+, $|z| = R$, and so $|z^4| = R^4$, so

$$R^4 - 1 \leq |z^4 + 1| \leq R^4 + 1$$

so

$$\frac{1}{R^4+1} \leq |\frac{1}{z^4+1}| \leq \frac{1}{R^4-1}.$$

The contour C_R^+ has length πR, so we have

$$|\int_{C_R^+} \frac{dz}{z^4+1}| \leq \frac{\pi R}{R^4-1}.$$

The right-hand side goes to zero as $R \to \infty$, and so by the squeeze theorem the left-hand side must also go to zero. Thus we have

$$\int_{-\infty}^{\infty} \frac{dx}{x^4+1} = \frac{\pi}{\sqrt{2}}.$$

Again, note that the final answer is a real number, as it ought to be.

- By symmetry, we can also integrate this function on half-lines:

$$\int_0^{\infty} \frac{dx}{x^4+1} = \frac{\pi}{2\sqrt{2}}.$$

- This basic strategy works whenever one is integrating a rational function $P(x)/Q(x)$, where P and Q are polynomials, from $-\infty$ to ∞ - provided that Q has a somewhat larger degree than P. By repeating the above argument one can show

- **Lemma.** If P and Q are polynomials such that $deg(Q) \geq deg(P)+2$, then $\int_{C_R^+} \frac{P(z)}{Q(z)}\, dz \to 0$ as $R \to \infty$.

- This lemma is also valid for the lower semi-circle C_R^-.

- If $deg(Q) < deg(P)+2$, then $\frac{P(z)}{Q(z)}$ is not integrable on the real line (it decays too slowly). For instance, $\int_{-\infty}^{\infty} \frac{x^2 dx}{x^2+1}$ does not converge (at infinity, it's like integrating the function 1).

Integrating trigonometric-rational functions

- Now let's try the same technique to integrate

$$\int_{-\infty}^{\infty} \frac{e^{-ix}}{x^2+1}\, dx.$$

- We can write this, as before, as

$$\lim_{R\to\infty} \int_{\gamma_R} \frac{e^{-iz}}{z^2+1}\, dz$$

- We can compute

$$\int_{\gamma_R + -C_R^+} \frac{e^{-iz}}{z^2+1}\, dz$$

using residue calculus; in this case, it turns out to equal πe. So to follow the previous pattern, we now just have to show that the C_R^+ integral goes to zero.

- Unfortunately, there is now a snag, because

$$\int_{C_R^+} \frac{e^{-iz}\, dz}{z^2+1}$$

does not go to zero as $R \to \infty$. The problem is that e^{-iz} can get extremely large on C_R^+. For instance, if $z = iR$, which is on C_R^+, then $e^{-iz} = e^R$, which becomes very large as $R \to \infty$.

- To get around this problem we need to use the *lower semi-circle* C_R^- from $-R$ to R rather than the upper semi-circle. To make sure that this will work, let us first check that

$$\int_{C_R^-} \frac{e^{-iz}\, dz}{z^2+1} \to 0 \text{ as } R \to \infty.$$

- If z is on C_R^-, then $|z| = R$, so $|z^2| = R^2$, so

$$R^2 - 1 \leq |z^2+1| \leq R^2 + 1.$$

What about e^{-iz}? We use Cartesian co-ordinates $z = x + iy$, and compute

$$|e^{-iz}| = |e^{-i(x+iy)}| = |e^{-ix}e^y| = e^y.$$

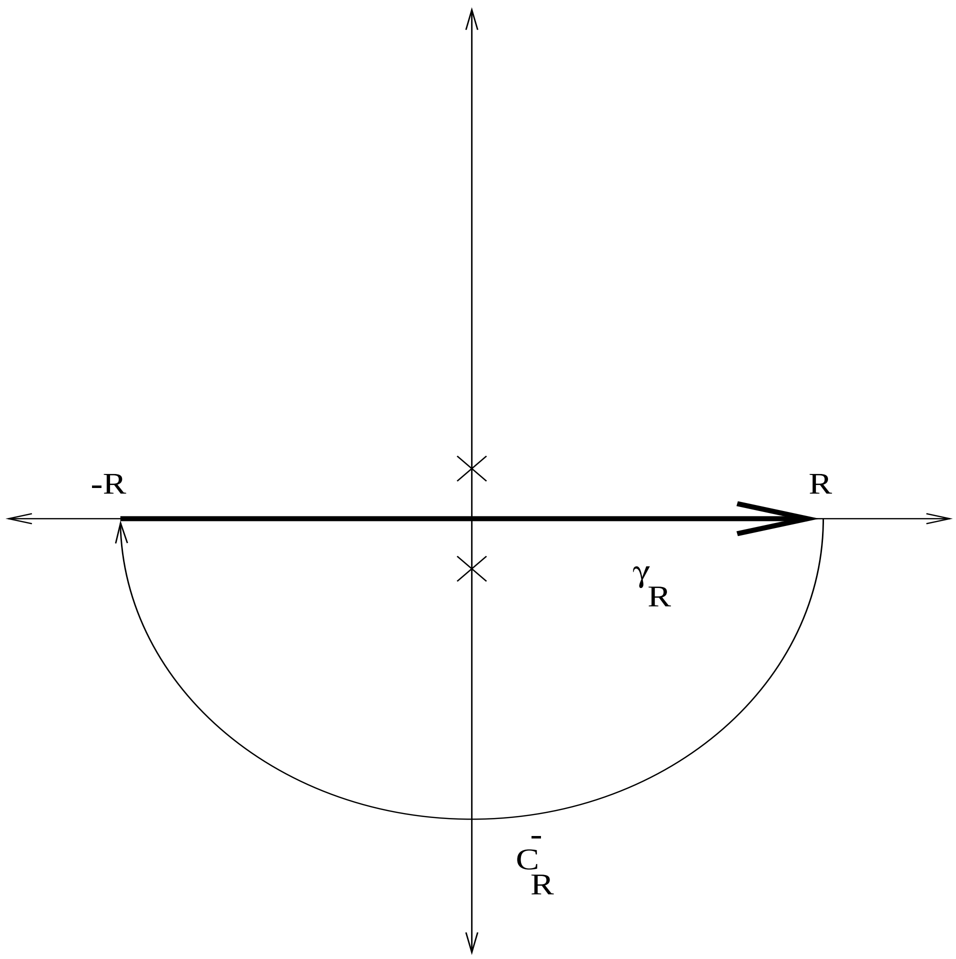
However, since we are on the *lower* semi-circle, $y \leq 0$, and so $e^y \leq 1$. Thus the numerator has magnitude at most 1, and the denominator has magnitude at least $R^2 - 1$, so

$$\left|\frac{e^{-iz}}{z^2+1}\right| \leq \frac{1}{R^2-1}.$$

Since C_R^- has length πR, we thus have

$$|\int_{C_R^-} \frac{e^{-iz}\ dz}{z^2+1}| \leq \frac{\pi R}{R^2-1}.$$

Thus the integral on the left-hand side goes to zero as $R \to \infty$.



- We now compute the closed contour integral

$$\oint_{\gamma_R + -C_R^-} \frac{e^{-iz}\ dz}{z^2+1}.$$

Let $f(z) = \frac{e^{-iz}}{z^2+1}$. The function f has singularities at $z = \pm i$, but only the singularity at $-i$ is relevant. Since $\gamma_R + -C_R^-$ winds once *clockwise* around $-i$, we have

$$\oint_{\gamma_R + -C_R^-} \frac{e^{-iz}\ dz}{z^2+1} = -2\pi i Res(f;-i).$$

- The function z^2+1 has a simple zero at $-i$, and e^{-iz} is non-zero at $-i$, so $f(z)$ has a simple pole, so

$$Res(f;-i) = \lim_{z\to -i} \frac{e^{-iz}(z+i)}{z^2+1}$$
$$= \lim_{z\to -i} \frac{e^{-iz}}{z-i} = \frac{e^{-1}}{-2i} = \frac{-1}{2ei}.$$

Thus we have

$$\oint_{\gamma_R + -C_R^-} \frac{e^{-iz}\ dz}{z^2+1} = \frac{\pi}{e}.$$

- Now we take limits as $R \to \infty$. We've already observed that the C_R integral goes to zero, so we're left with

$$\int_{-\infty}^{\infty} \frac{e^{-ix}\ dx}{x^2+1} = \frac{\pi}{e}.$$

- In general, when integrating an expression such as

$$\int_{-\infty}^{\infty} \frac{e^{imx}P(x)}{Q(x)}\ dx,$$

where P and Q are polynomials and m is a non-zero constant, then one should use the upper semi-circular contour when $m > 0$ and the lower semi-circular contour when $m < 0$. (Basically, the idea is to push the contour to wherever the integrand is small). There is a general lemma:

- **Jordan's lemma:** Let P and Q be polynomials such that $deg(Q) \geq deg(P) + 1$, and let m be a non-zero number. If $m > 0$, then

$$\int_{C_R^+} \frac{e^{imz}P(z)}{Q(z)}\ dz \to 0 \text{ as } R \to \infty;$$

and if $m < 0$, then

$$\int_{C_R^-} \frac{e^{imz}P(z)}{Q(z)}\ dz \to 0 \text{ as } R \to \infty;$$

- The proof of this fact is a bit tedious, and will be omitted. (One has to divide the contour into two pieces, one when $Im(z)$ is small, say $|Im(z)| \leq \sqrt{R}$, and the other when $Im(z)$ is large). Note that Q only needs to have a degree one higher than P, as opposed to two higher for the previous lemma. That's because the e^{imz} factor helps to make the integrand smaller. (If Q has a degree equal to or less than P, then the integral is not convergent).

- One can also handle trigonometric-rational expressions such as

$$\int_{-\infty}^{\infty} \frac{\cos(x)}{x^2+1}\, dx$$

in this manner. One way to do this is to use the formula

$$\cos(x) = \frac{e^{ix} + e^{-ix}}{2}$$

to split this integral into two pieces,

$$\frac{1}{2}\int_{-\infty}^{\infty} \frac{e^{ix}}{x^2+1}\, dx + \frac{1}{2}\int_{-\infty}^{\infty} \frac{e^{-ix}}{x^2+1}\, dx$$

and work on each piece separately (the first one using the upper semi-circle, the second one using the lower semi-circle). But this takes quite a while. A quicker way, in this case, is just to observe that $\cos(x)$ is the real part of e^{-ix} (or e^{ix}), so $\frac{\cos(x)}{x^2+1}$ is the real part of $\frac{e^{-ix}}{x^2+1}$. Thus

$$\int_{-\infty}^{\infty} \frac{\cos(x)}{x^2+1}\, dx = Re \int_{-\infty}^{\infty} \frac{e^{-ix}}{x^2+1}\, dx = \frac{\pi}{e}.$$

Similarly we have

$$\int_{-\infty}^{\infty} \frac{\sin(x)}{x^2+1}\, dx = -Im \int_{-\infty}^{\infty} \frac{e^{-ix}}{x^2+1}\, dx = 0;$$

although this is more easily proven by observing that $\frac{\sin(x)}{x^2+1}$ is an odd function. By symmetry we also have

$$\int_{0}^{\infty} \frac{\cos(x)}{x^2+1}\, dx = \frac{\pi}{2e}.$$

- However, one has to be a bit careful when dealing with complex expressions such as
$$\int_{-\infty}^{\infty} \frac{\cos(x)}{x+i}\, dx.$$
Even though $\cos(x)$ is the real part of $e^{\pm ix}$, the integrand $\frac{\cos(x)}{x+i}$ is *not* the real part of $\frac{e^{\pm ix}}{x+i}$, because $x+i$ itself has real and imaginary parts. (The real part of $\frac{e^{ix}}{x+i}$ is something much messier, namely $\frac{x\cos(x)+\sin(x)}{x^2+1}$). In this case the best thing to do is use the alternate approach above, namely splitting $\cos(x) = \frac{1}{2}e^{ix} + \frac{1}{2}e^{-ix}$ and working on both the upper and lower semi-circles.

Principal value integrals

- Now we turn to integrals which are not, strictly speaking, convergent, but which can still be evaluated in what's called a "principal value" sense. These integrals crop up every now and then, for instance in the boundary behaviour of PDE.

- As an example, consider the real integral
$$\int_{-1}^{2} \frac{dx}{x}.$$

- Naively, one would expect this integral to equal
$$\ln|x||_{-1}^{2} = \ln(2) - \ln(1) = \ln(2).$$

- However, strictly speaking, $\frac{1}{x}$ is not integrable on the interval $[-1,2]$ because of the singularity at 0. (The fundamental theorem of calculus does not apply on $[-1,2]$ for the same reason). There is an infinite amount of area in the first and fourth quadrants, and the expression $\infty - \infty$ is indefinite. Because of the singularity, this type of integral is sometimes called a *singular integral.*

- Despite singular integrals being non-convergent, there is still a certain sense in which the integral of $\frac{1}{x}$ from -1 to 2 equals $\ln(2)$. We define the *principal value* of a singular integral to equal the limit, as $\varepsilon \to 0$, of the integral where an interval of radius ε around the singularity has been removed. In this case, we have

$$p.v. \int_{-1}^{2} \frac{dx}{x} = \lim_{\varepsilon \to 0} \int_{-1}^{-\varepsilon} \frac{dx}{x} + \int_{\varepsilon}^{2} \frac{dx}{x}.$$

- Since we have cut out the singularity, the right-hand side is computable as

$$\lim_{\varepsilon \to 0} \ln |x| |_{-1}^{-\varepsilon} + \ln |x| |_{\varepsilon}^{2}$$
$$= \lim_{\varepsilon \to 0} \ln \varepsilon - \ln 1 + \ln 2 - \ln \varepsilon = \ln 2.$$

  Thus the principal value of $\int_{-1}^{2} \frac{dx}{x} = \ln 2$.

- This principal value definition is more subtle than it may first appear. It is important that the interval one removes around the singularity is symmetric. Suppose we removed the interval $(-\varepsilon, 2\varepsilon)$ from the origin instead of $(-\varepsilon, \varepsilon)$, and then let $\varepsilon \to 0$. We would then get

$$\lim_{\varepsilon \to 0} \ln |x| |_{-1}^{-\varepsilon} + \ln |x| |_{2\varepsilon}^{2}$$

$$= \lim_{\varepsilon \to 0} \ln \varepsilon - \ln 1 + \ln 2 - \ln 2\varepsilon = 0!$$

- If there are multiple singularities, we remove an epsilon interval around all of them simultaneously, and then let epsilon go to zero.

Computing principal value integrals using residues

- We now show how to compute principal value integrals using residue calculus, using what's known as the method of indented contours. As an example, we demonstrate the computation of

$$p.v. \int_{-\infty}^{\infty} \frac{e^{ix}\, dx}{x^3 + x}.$$

  Note that this is a singular integral because the integrand has a singularity at $x = 0$.

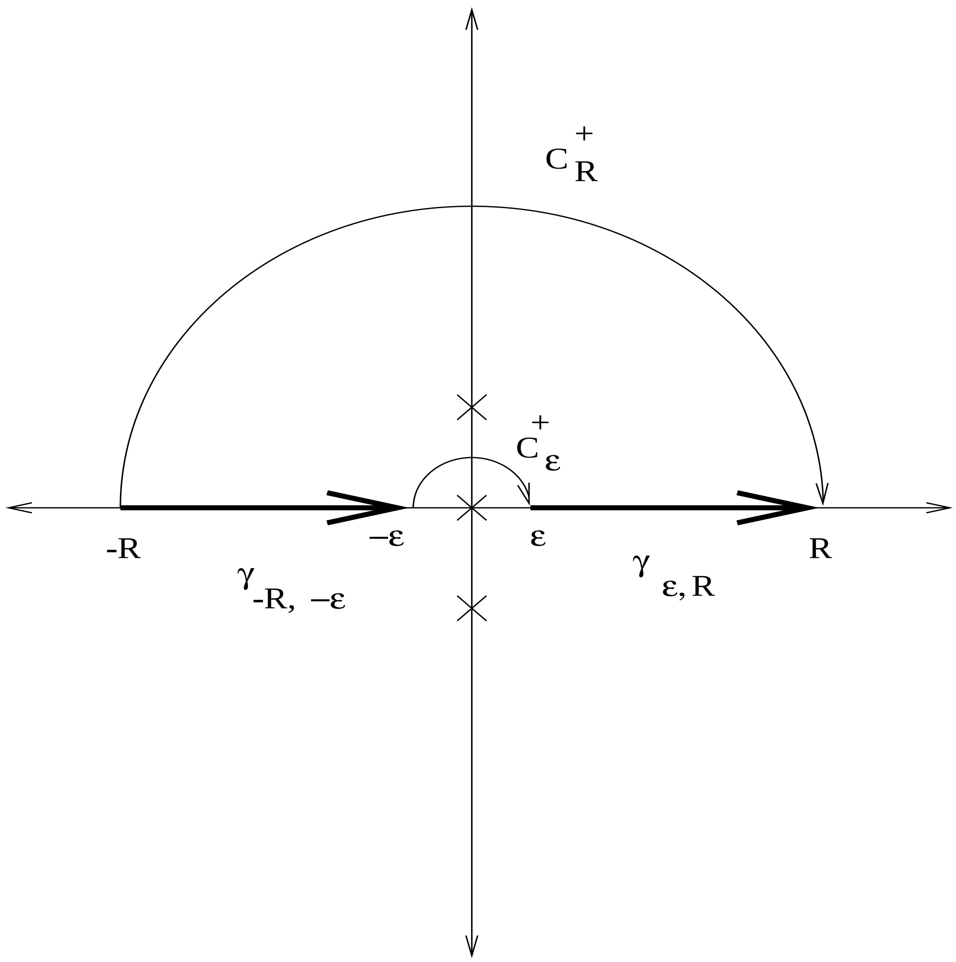
- If we didn't have the singularity, we'd just replace the domain of integration from $(-\infty, \infty)$ to $(-R, R)$ and then let $R \to \infty$ as before. We can still do that, but now we must also remove an ε around the singularity and let $\varepsilon \to 0$. In other words, we can rewrite the above integral as

$$\lim_{R\to\infty} \lim_{\varepsilon\to 0} \int_{-R}^{-\varepsilon} \frac{e^{ix}\ dx}{x^3+x} + \int_{\varepsilon}^{R} \frac{e^{ix}\ dx}{x^3+x}.$$

  We can rewrite this as

$$\lim_{R\to\infty} \lim_{\varepsilon\to 0} \int_{\gamma_{-R,-\varepsilon}} f(z)\ dz + \int_{\gamma_{\varepsilon,R}} f(z)\ dz$$

  where $f(z) = \frac{e^{iz}}{z^3+z}$, $\gamma_{-R,-\varepsilon}$ is the straight line from $-R$ to $-\varepsilon$, and $\gamma_{\varepsilon,R}$ is the straight line from ε to R.



- To use the residue theorem we need to find a closed contour somewhere. By adding the big semi-circle C_R^+ one can partially close the contour, but to make a genuinely closed contour one also needs the small semi-circle C_ε^+. (One could also use the lower semi-circle C_ε^- instead of C_ε^+, but one has to use C_R^+ and not C_R^- because otherwise e^{iz} will become far too large.

- We can compute the closed contour integral

$$\oint_{\gamma_{-R,-\varepsilon}+C_\varepsilon^+ + \gamma_{\varepsilon,R} + -C_R^+} f(z)\ dz$$

using residue calculus. Factoring f as

$$f(z) = \frac{e^{iz}}{z(z+i)(z-i)}$$

we see that f has simple poles at 0, $-i$, and i. Of these, only the singularity at i is inside the contour. Since the contour is anti-clockwise, we see that the above integral is equal to

$$2\pi i Res(f; i) = 2\pi i \lim_{z \to i} \frac{e^{iz}(z-i)}{z(z+i)(z-i)}$$
$$= 2\pi i \frac{e^{-1}}{i(i+i)} = \frac{-\pi i}{e}.$$

Thus

$$\int_{\gamma_{-R,-\varepsilon}} f(z)\ dz + \int_{C_\varepsilon^+} f(z)\ dz + \int_{\gamma_{\varepsilon,R}} f(z)\ dz - \int_{C_R^+} f(z)\ dz = \frac{-\pi i}{e}.$$

Now we take limits as $R \to \infty$ and $\varepsilon \to 0$. The first and third terms are what we want. The fourth term goes to zero by Jordan's lemma. Thus we have

$$p.v. \int_{-\infty}^{\infty} \frac{e^{ix}\ dx}{x^3 + x} + \lim_{\varepsilon \to 0} \int_{C_\varepsilon^+} f(z)\ dz = \frac{-\pi i}{e}.$$

- To finish our computation we need to figure out what $\int_{C_\varepsilon^+} f(z)\ dz$.
- If we were integrating over the entire clockwise circle C_ε of radius ε, instead of just a clockwise semi-circle C_ε^+, then this integral would equal $2\pi i Res(f; 0)$ by the residue theorem. Since we only have half of the contour, one might guess that the integral on the semi-circle should be approximately $\pi i Res(f; 0)$. This is indeed the case:
- **Lemma.** Let $f(z)$ be a function with a simple pole at z_0, and let $\theta_1 \leq \theta_2$ be angles. For each ε, let γ_ε denote the anti-clockwise $z(t) = z_0 + \varepsilon e^{i\theta}$, $\theta_1 \leq \theta \leq \theta_2$. Then

$$\lim_{\varepsilon \to 0} \int_{\gamma_\varepsilon} f(z)\ dz = (\theta_2 - \theta_1) i Res(f; z_0).$$

- **Proof** Since $f(z)$ has a simple pole at z_0, we can write $f(z) = g(z)/(z-z_0)$, where g is a function which is non-zero and analytic at z_0. From Taylor's formula

$$g(z) = g(z_0) + g'(z_0)(z - z_0) + \dots$$

we have the Laurent expansion

$$f(z) = \frac{g(z)}{z - z_0} = \frac{g(z_0)}{z - z_0} + g'(z_0) + \dots$$

and so $Res(f; z_0) = g(z_0)$. So we have to show that

$$\lim_{\varepsilon \to 0} \int_{\gamma_\varepsilon} \frac{g(z)}{z - z_0}\ dz = (\theta_2 - \theta_1) i g(z_0).$$

To do this, we make the change of variables $z = z_0 + \varepsilon e^{i\theta}$, $dz = \varepsilon i e^{i\theta} d\theta$, to turn the left-hand side into

$$\lim_{\varepsilon \to 0} \int_{\theta_1}^{\theta_2} \frac{g(z_0 + \varepsilon e^{i\theta})}{\varepsilon e^{i\theta}}\ \varepsilon i e^{i\theta} d\theta$$

which simplifies to

$$i \lim_{\varepsilon \to 0} \int_{\theta_1}^{\theta_2} g(z_0 + \varepsilon e^{i\theta})\ d\theta.$$

Since g is analytic at z_0, it is continuous, and so $g(z_0 + \varepsilon e^{i\theta}) \to g(z_0)$ as $\varepsilon \to 0$. Taking limits (cf. the proof of Cauchy-Goursat in Week 5) the left-hand side becomes

$$i \int_{\theta_1}^{\theta_2} g(z_0)\ d\theta$$

which equals the right-hand side.

- In our case, we have a clockwise semi-circle from angles π to 0, but we can just flip the sign and consider the reversed anti-clockwise circle $-C_\varepsilon^+$, which goes from angles 0 to π. By the lemma,

$$\lim_{\varepsilon \to 0} \int_{-C_\varepsilon^+} f(z)\ dz = \pi i Res(f; 0)$$

$$= \pi i \lim_{z\to 0} z \frac{e^{iz}}{z^3+z}$$
$$= \pi i.$$

Putting this back into our previous calculations, we get

$$p.v. \int_{-\infty}^{\infty} \frac{e^{ix}\ dx}{x^3+x} - \pi i = \frac{-\pi i}{e},$$

thus

$$p.v. \int_{-\infty}^{\infty} \frac{e^{ix}\ dx}{x^3+x} = \pi(1-\frac{1}{e})i.$$

- As always, one can take real and imaginary parts to conclude

$$p.v. \int_{-\infty}^{\infty} \frac{\cos(x)\ dx}{x^3+x} = 0$$

$$p.v. \int_{-\infty}^{\infty} \frac{sin(x)\ dx}{x^3+x} = \pi(1-\frac{1}{e});$$

since the latter integral is even, we have

$$p.v. \int_{0}^{\infty} \frac{sin(x)\ dx}{x^3+x} = \frac{1}{2}\pi(1-\frac{1}{e}).$$