Math 131BH - Week 6
Textbook pages covered: 185-192

- Periodic functions
- Inner products, the L^2 metric, and convolution for periodic functions
- Trigonometric polynomials
- Approximation by trigonometric polynomials
- Fourier series; Fourier and Plancherel theorems

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Overview of Fourier series

- In the last two weeks, we discussed the issue of how certain functions (for instance, compactly supported continuous functions) could be approximated by polynomials. Later, we showed how a different class of functions (real analytic functions) could be written exactly (not approximately) as an infinite polynomial, or more precisely a power series.
- Power series are already immensely useful, especially when dealing with special functions such as the exponential and trigonometric functions discussed earlier. However, there are some circumstances where power series are not so useful, because one has to deal with functions (e.g. $\sqrt{x}$) which are not real analytic, and so which do not have power series.
- Fortunately, there is another type of series expansion, known as *Fourier series*, which is also a very powerful tool in analysis (though used for slightly different purposes). Instead of analyzing compactly supported functions, it instead analyzes *periodic functions*; instead of decomposing into polynomials, it decomposes into *trigonometric polynomials*. Roughly speaking, the theory of Fourier series asserts that just about every periodic function can be decomposed as an (infinite) sum of sines and cosines.

- A historical note: Jean-Baptiste Fourier (1768-1830) was, among other things, the governor of Egypt during the reign of Napoleon. After the Napoleonic wars, he returned to mathematics. He introduced Fourier series in an important 1807 paper in which he used them to solve what is now known as the heat equation. At the time, the claim that every periodic function could be expressed as a sum of sines and cosines was extremely controversial, even such leading mathematicians as Euler declared that it was impossible. Nevertheless, Fourier managed to show that this was indeed the case, although the proof was not completely rigorous and was not totally accepted for almost another hundred years.

- There will be some similarities between the theory of Fourier series and that of power series, but there are also some major differences. For instance, the convergence of Fourier series is usually not uniform (i.e. not in the L^∞ metric), but instead we have convergence in a different metric, the L^2-metric. Also, we will need to use complex numbers heavily in our theory, while they played only a tangential role in power series.

- The theory of Fourier series (and of related topics such as Fourier integrals and the Laplace transform) is vast, and deserves an entire course in itself. (At UCLA, that course is Math 133, Introduction to Fourier Analysis). It has many, many applications, most directly to differential equations, signal processing, electrical engineering, physics, and analysis, but also to algebra and number theory. We will only give the barest bones of the theory here, however, and almost no applications.

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Periodic functions

- The theory of Fourier series has to do with the analysis of (complex-valued) *periodic functions*, which we now define.

- **Definition** Let $L > 0$ be a real number. A function $f : \mathbf{R} \to \mathbf{C}$ is *periodic with period L*, or *L-periodic*, if we have $f(x + L) = f(x)$ for every real number x.

- **Example** The real-valued functions $f(x) = \sin(x)$ and $f(x) = \cos(x)$ are 2π-periodic, as is the complex-valued function $f(x) = e^{ix}$. These functions are also 4π-periodic, 6π-periodic, etc. (why?). The function $f(x) = x$, however, is not periodic. The constant function $f(x) = 1$ is L-periodic for every L.

- If a function f is L-periodic, then we have $f(x + kL) = f(x)$ for every integer k (why? Use induction for the positive k, and then use a substitution to convert the positive k result to a negative k result. The $k = 0$ case is of course trivial). In particular, if a function f is 1-periodic, then we have $f(x + k) = f(x)$ for every $k \in \mathbf{Z}$. Because of this, 1-periodic functions are sometimes also called $\mathbf{Z}$-*periodic* (and L-periodic functions called $L\mathbf{Z}$-periodic).

- **Example** For any integer n, the functions $\cos(2\pi nx)$, $\sin(2\pi nx)$, and $e^{2\pi inx}$ are all $\mathbf{Z}$-periodic. (What happens when n is not an integer?). Another example of a $\mathbf{Z}$-periodic function is the function $f : \mathbf{R} \to \mathbf{C}$ defined by $f(x) := 1$ when $x \in [n, n + \frac{1}{2})$ for some integer n, and $f(x) := 0$ when $x \in [n + \frac{1}{2}, n + 1)$ for some integer n. This function is an example of a *square wave*.

- Henceforth, for simplicity, we shall only deal with functions which are $\mathbf{Z}$-periodic (for the general theory of L-periodic functions, see the homework). Note that in order to completely specify a $\mathbf{Z}$-periodic function $f : \mathbf{R} \to \mathbf{C}$, one only needs to specify its values on the interval $[0, 1)$, since this will determine the values of f everywhere else. This is because every real number x can be written in the form $x = k + y$ where k is an integer (called the *integer part* of x, and sometimes denoted $[x]$) and $y \in [0, 1)$ (this is called the *fractional part* of x, and sometimes denoted $\{x\}$). (For those of you who have seen the construction of the real numbers in 131AH, you can challenge yourself to actually prove that this decomposition $x = k + y$ exists and is unique). Because of this, sometimes when we wish to describe a $\mathbf{Z}$-periodic function f we just describe what it does on the interval $[0, 1)$, and then say that it is *extended periodically* to all of $\mathbf{R}$. This means that we define $f(x)$ for any real number x by setting $f(x) := f(y)$, where we have decomposed $x = k + y$ as discussed above. (One can in fact replace the interval $[0, 1)$ by any other half-open interval of length 1, but we will not do so here).

- The space of complex-valued continuous $\mathbf{Z}$-periodic functions is denoted $C(\mathbf{R}/\mathbf{Z};\mathbf{C})$. (The notation $\mathbf{R}/\mathbf{Z}$ comes from algebra, and denotes the quotient group of the additive group $\mathbf{R}$ by the additive group $\mathbf{Z}$; more information in this can be found in Math 110B). By "continuous" we mean continuous at all points on $\mathbf{R}$; merely being continuous on an interval such as $[0,1]$ will not suffice, as there may be a discontinuity between the left and right limits at 1 (or at any other integer). Thus for instance, the functions $\sin(2\pi nx)$, $\cos(2\pi nx)$, and $e^{2\pi inx}$ are all elements of $C(\mathbf{R}/\mathbf{Z};\mathbf{C})$, as are the constant functions, however the square wave function described earlier is not in $C(\mathbf{R}/\mathbf{Z};\mathbf{C})$ because it is not continuous. Also the function $\sin(x)$ would also not qualify to be in $C(\mathbf{R}/\mathbf{Z};\mathbf{C})$ since it is not $\mathbf{Z}$-periodic.

- A couple basic properties of $C(\mathbf{R}/\mathbf{Z};\mathbf{C})$.

- **Lemma 1**

- (i) (Continuous periodic functions are bounded) If $f \in C(\mathbf{R}/\mathbf{Z};\mathbf{C})$, then f is bounded (i.e. there exists a real number $M > 0$ such that $|f(x)| \leq M$ for all $x \in \mathbf{R}$).

- (ii) (Continuous periodic functions form a vector space and an algebra) If $f, g \in C(\mathbf{R}/\mathbf{Z};\mathbf{C})$, then the functions $f + g$, $f - g$, and fg are also in $C(\mathbf{R}/\mathbf{Z};\mathbf{C})$. Also, if c is any complex number, then the function cf is also in $C(\mathbf{R}/\mathbf{Z};\mathbf{C})$.

- (iii) (Uniform limit of continuous periodic functions is continuous periodic) If $(f_n)_{n=1}^{\infty}$ is a sequence of functions in $C(\mathbf{R}/\mathbf{Z};\mathbf{C})$ which converges uniformly to another function $f : \mathbf{R} \to \mathbf{C}$, then f is also in $C(\mathbf{R}/\mathbf{Z};\mathbf{C})$.

- **Proof.** See Week 6 homework. $\square$

- One can make $C(\mathbf{R}/\mathbf{Z};\mathbf{C})$ into a metric space by re-introducing the now familiar sup-norm metric

$$d_\infty(f,g) = \sup_{x\in\mathbf{R}} |f(x) - g(x)| = \sup_{x\in[0,1)} |f(x) - g(x)|$$

  of uniform convergence. (Why is the first supremum the same as the second?). Using this metric, one can show that $C(\mathbf{R}/\mathbf{Z};\mathbf{C})$ is a complete metric space, but we will not need to do so here.

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Inner products on periodic functions

- From Lemma 1 we know that we can add, subtract, multiply, and take limits of continuous periodic functions. We will need a couple more operations on the space $C(\mathbf{R}/\mathbf{Z};\mathbf{C})$, though. The first one is that of *inner product.*

- **Definition** If $f, g \in C(\mathbf{R}/\mathbf{Z};\mathbf{C})$, we define the *inner product* $\langle f, g\rangle$ to be the quantity
$$\langle f, g\rangle = \int_{[0,1]} f(x)\overline{g(x)}\ dx.$$

- (Note: in order to integrate a complex-valued function, $f(x) = g(x) + ih(x)$, we use the definition that $\int_{[a,b]} f := \int_{[a,b]} g + i\int_{[a,b]} h$; i.e. we integrate the real and imaginary parts of the function separately. Thus for instance $\int_{[1,2]}(1 + ix)\ dx = \int_{[1,2]} 1\ dx + i\int_{[1,2]} x\ dx = 1 + \frac{3}{2}i$. It is easy to verify that all the standard rules of calculus (integration by parts, fundamental theorem of calculus, substitution, etc.) still hold when the functions are complex-valued instead of real-valued).

- **Example** Let f be the constant function $f(x) := 1$, and let $g(x)$ be the function $g(x) := e^{2\pi ix}$. Then we have
$$\langle f, g\rangle = \int_{[0,1]} 1\overline{e^{2\pi ix}}\ dx = \int_{[0,1]} e^{-2\pi ix}\ dx = \frac{e^{-2\pi ix}}{-2\pi i}|_{x=0}^{x=1} = \frac{e^{-2\pi i} - e^0}{-2\pi i} = \frac{1-1}{-2\pi i} = 0.$$

- In general, the inner product $\langle f, g\rangle$ will be a complex number. (Note that $f(x)\overline{g(x)}$ will be Riemann integrable since both functions are bounded and continuous.)

- Roughly speaking, the inner product is to the space $C(\mathbf{R}/\mathbf{Z};\mathbf{C})$ as the dot product $x \cdot y$ is to Euclidean spaces such as $\mathbf{R}^n$; see Math 115A for more details on this. We list some basic properties of the inner product below (for a more in-depth study of inner products on vector spaces, see Math 115A).

- **Lemma 2.** Let $f, g, h \in C(\mathbf{R}/\mathbf{Z};\mathbf{C})$.

- (i) (Hermitian property) We have $\langle g, f\rangle = \overline{\langle f, g\rangle}$.
- (ii) (Positivity) The number $\langle f, f\rangle$ is real and non-negative: $\langle f, f\rangle \geq 0$. We have $\langle f, f\rangle = 0$ if and only if $f = 0$ (i.e. $f(x) = 0$ for all $x \in \mathbf{R}$).
- (iii) (Linearity in the first variable) We have $\langle f+g, h\rangle = \langle f, h\rangle + \langle g, h\rangle$. For any complex number c, we have $\langle cf, g\rangle = c\langle f, g\rangle$.
- (iv) (Antilinearity in the second variable) We have $\langle f, g+h\rangle = \langle f, g\rangle + \langle f, h\rangle$. For any complex number c, we have $\langle f, cg\rangle = \overline{c}\langle f, g\rangle$.
- **Proof.** See Week 6 homework. $\square$
- From the positivity property, it makes sense to define the L^2 *norm* $\|f\|_2$ of a function $f \in C(\mathbf{R}/\mathbf{Z};\mathbf{C})$ by the formula

$$\|f\|_2 := \sqrt{\langle f, f\rangle} = (\int_{[0,1]} f(x)\overline{f(x)}\ dx)^{1/2} = (\int_{[0,1]} |f(x)|^2\ dx)^{1/2}.$$

  Thus $\|f\|_2 \geq 0$ for all f. The norm $\|f\|_2$ is sometimes called the *root mean square* of f.
- **Example** If $f(x)$ is the function $e^{2\pi i x}$, then

$$\|f\|_2 = (\int_{[0,1]} e^{2\pi i x}e^{-2\pi i x}\ dx)^{1/2} = (\int_{[0,1]} 1\ dx)^{1/2} = 1^{1/2} = 1.$$

- This L^2 norm is related to, but is distinct from, the L^∞ norm $\|f\|_\infty := \sup_{x\in\mathbf{R}} |f(x)|$. For instance, if $f(x) = \sin(x)$, then $\|f\|_\infty = 1$ but $\|f\|_2 = \frac{1}{\sqrt{2}}$. In general we always have the inequality $\|f\|_2 \leq \|f\|_\infty$ (why?).
- Some basic properties of the L^2 norm are given below.
- **Lemma 3.** Let $f, g \in C(\mathbf{R}/\mathbf{Z};\mathbf{C})$.
- (i) (Non-degeneracy) We have $\|f\|_2 = 0$ if and only if $f = 0$.
- (ii) (Cauchy-Schwarz inequality) We have $|\langle f, g\rangle| \leq \|f\|_2\|g\|_2$.
- (iii) (Triangle inequality) We have $\|f+g\|_2 \leq \|f\|_2 + \|g\|_2$.

- (iv) (Pythagoras theorem) If $\langle f, g \rangle = 0$, then $\|f+g\|_2^2 = \|f\|_2^2 + \|g\|_2^2$.
- (v) (Homogeneity) If c is a complex number, then $\|cf\|_2 = |c|\|f\|_2$.
- **Proof.** See Week 6 homework. $\square$
- In light of Pythagoras's theorem, we sometimes say that f and g are *orthogonal* iff $\langle f, g \rangle = 0$.
- We can now define the L^2 *metric* d_{L^2} on $C(\mathbf{R}/\mathbf{Z}; \mathbf{C})$ by defining

$$d_{L^2}(f, g) := \|f - g\|_2 = (\int_{[0,1]} |f(x) - g(x)|^2\ dx)^{1/2}.$$

  One can easily verify that d_{L^2} is indeed a metric (this is similar to Q2(b) of Assignment 1). Indeed, the L^2 metric is very similar to the l^2 metric on Euclidean spaces $\mathbf{R}^n$, which is why the notation is deliberately chosen to be similar; you should compare the two metrics yourself to see the analogy.

- Note that a sequence f_n of functions in $C(\mathbf{R}/\mathbf{Z}; \mathbf{C})$ will *converge in the L^2 metric* to $f \in C(\mathbf{R}/\mathbf{Z}; \mathbf{C})$ if $d_{L^2}(f_n, f) \to 0$ as $n \to \infty$, or in other words that

$$\lim_{n\to\infty} \int_{[0,1]} |f_n(x) - f(x)|^2\ dx = 0.$$

- The notion of convergence in L^2 metric is different from that of uniform or pointwise convergence; see homework.
- The L^2 metric is not as well-behaved as the L^∞ metric. For instance, it turns out the space $C(\mathbf{R}/\mathbf{Z}; \mathbf{C})$ is not complete in the L^2 metric, despite being complete in the L^∞ metric; for instance, it is easy to find a sequence of continuous periodic functions which converge in L^2 to a discontinuous periodic function. (Can you think of one? Try converging to the square wave function).

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Trigonometric polynomials

- We now define the concept of a *trigonometric polynomial.* Just as polynomials are combinations of the functions x^n (sometimes called *monomials*), trigonometric polynomials are combinations of the functions $e^{2\pi inx}$ (sometimes called *characters*).

- **Definition** For every integer n, we let $e_n \in C(\mathbf{R}/\mathbf{Z}; \mathbf{C})$ denote the function
$$e_n(x) := e^{2\pi inx}.$$
This is sometimes referred to as the *character with frequency n.*

- **Definition** A function $f \in C(\mathbf{R}/\mathbf{Z}; \mathbf{C})$ is said to be a *trigonometric polynomial* if we can write $f = \sum_{n=-N}^{N} c_n e_n$ for some integer $N \geq 0$ and some complex numbers $(c_n)_{n=-N}^{N}$.

- **Example.** The function $f = 4e_{-2} + ie_{-1} - 2e_0 + 0e_1 - 3e_2$ is a trigonometric polynomial; it can be written more explicitly as
$$f(x) = 4e^{-4\pi ix} + ie^{-2\pi ix} - 2 - 3e^{4\pi ix}.$$

- **Example.** For any integer n, the function $f(x) := \cos(2\pi nx)$ is a trigonometric polynomial, since $\cos(2\pi nx) = \frac{e^{2\pi inx}+e^{-2\pi inx}}{2}$ and thus $f = \frac{1}{2}e_{-n} + \frac{1}{2}e_n$. Similarly the function $f(x) := \sin(2\pi nx)$ is a trigonometric polynomial since $f = \frac{-1}{2i}e_{-n} + \frac{1}{2i}e_n$. In particular, any linear combination of sines and cosines is also a trigonometric polynomial, for instance $f(x) = 3 + i\cos(2\pi x) + 4i\sin(4\pi x)$ is a trigonometric polynomial.

- The Fourier theorem will allow us to write any function in $C(\mathbf{R}/\mathbf{Z}; \mathbf{C})$ as a Fourier series, which is to trigonometric polynomials as power series is to polynomials. To do this we will use the inner product structure from the previous section. The key computation is

- **Lemma 4.** (Characters are an orthonormal system) For any integers n and m, we have $\langle e_n, e_m \rangle = 1$ when $n = m$ and $\langle e_n, e_m \rangle = 0$ when $n \neq m$. Also, we have $\|e_n\| = 1$.

- **Proof.** See Week 6 homework. $\square$

- As a consequence, we have a formula for the co-efficients of a trigonometric polynomial.

- **Corollary 5.** Let $f = \sum_{n=-N}^{N} c_n e_n$ be a trigonometric polynomial. Then we have the formula

$$c_n = \langle f, e_n \rangle$$

for all integers $-N \leq n \leq N$. Also, we have $0 = \langle f, e_n \rangle$ whenever $n > N$ or $n < -N$. Also, we have the identity

$$\|f\|_2^2 = \sum_{n=-N}^{N} |c_n|^2.$$

- **Proof.** See Week 6 homework. $\square$

- We rewrite the conclusion of this corollary in a different way.

- **Definition** For any function $f \in C(\mathbf{R}/\mathbf{Z}; \mathbf{R})$, and any integer $n \in \mathbf{Z}$, we define the n^{th} *Fourier coefficient of* f, denoted $\hat{f}(n)$, by the formula

$$\hat{f}(n) := \langle f, e_n \rangle = \int_{[0,1]} f(x) e^{-2\pi i n x}\, dx.$$

The function $\hat{f} : \mathbf{Z} \to \mathbf{C}$ is called the *Fourier transform* of f.

- From Corollary 5, we thus see that whenever $f = \sum_{n=-N}^{N} c_n e_n$ is a trigonometric polynomial, we have

$$f = \sum_{n=-N}^{N} \langle f, e_n \rangle e_n = \sum_{n=-\infty}^{\infty} \langle f, e_n \rangle e_n$$

and in particular we have the *Fourier inversion formula*

$$f = \sum_{n=-\infty}^{\infty} \hat{f}(n) e_n$$

or in other words

$$f(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi i n x}.$$

Also, from the second identity of Corollary 5 we have the *Plancherel formula*

$$\|f\|_2^2 = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2.$$

- We stress that at present we have only proven the Fourier inversion and Plancherel formulae in the case when f is a trigonometric polynomial. Note that in this case that the Fourier coefficients $\hat{f}(n)$ are mostly zero (indeed, they can only be non-zero when $-N \leq n \leq N$), and so this infinite sum is really just a finite sum in disguise. In particular there are no issues about what sense the above series converge in; they both converge pointwise, uniformly, and in L^2 metric, since they are just finite sums.

- In the next few sections we will extend the Fourier inversion and Plancherel formulae to general functions in $C(\mathbf{R}/\mathbf{Z}; \mathbf{C})$, not just trigonometric polynomials. (It is also possible to extend the formula to discontinuous functions such as the square wave, but we will not do so here). To do this we will need a version of the Weierstrass approximation theorem, this time requiring that a continuous periodic function be approximated uniformly by *trigonometric* polynomials. Just as convolutions were used in the proof of the polynomial Weierstrass approximation theorem, we will also need a notion of convolution tailored for periodic functions.

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Periodic convolutions

- The goal of this section is to prove the

- **Weierstrass approximation theorem for trigonometric polynomials.** Let $f \in C(\mathbf{R}/\mathbf{Z}; \mathbf{C})$, and let $\varepsilon > 0$. Then there exists a trignometric polynomial P such that $\|f - P\|_\infty \leq \varepsilon$.

- In other words, any continuous periodic function can be uniformly approximated by trigonometric polynomials. To put it another way, if we let $P(\mathbf{R}/\mathbf{Z}; \mathbf{C})$ denote the space of all trigonometric polynomials, then the closure of $P(\mathbf{R}/\mathbf{Z}; \mathbf{C})$ in the L^∞ metric is $C(\mathbf{R}/\mathbf{Z}; \mathbf{C})$.

- It is possible to prove this theorem directly from the Weierstrass approximation theorem from the previous week's notes; however we shall instead prove this theorem from scratch, in order to introduce a couple interesting notions, notably that of periodic convolution. However, the proof here should strongly remind you of the arguments used to prove the other Weierstrass approximation theorem.

- **Definition** Let $f, g \in C(\mathbf{R}/\mathbf{Z}; \mathbf{C})$. Then we define the *periodic convolution* $f * g : \mathbf{R} \to \mathbf{C}$ of f and g by the formula

$$f * g(x) := \int_{[0,1]} f(y)g(x-y)\ dy.$$

- Note that this formula is slightly different from the convolution for compactly supported functions defined in the previous week's notes, because we are only integrating over $[0, 1]$ and not on all of $\mathbf{R}$. Thus, in principle we have given the symbol $f * g$ two conflicting meanings. However, in practice there will be no confusion, because it is not possible for a function to both be periodic and compactly supported (unless it is zero, but since $0 * f = f * 0 = 0$ under both definitions of convolution, there is still no confusion).

- Some basic properties of periodic convolution are as follows.

- **Lemma 6** Let $f, g, h \in C(\mathbf{R}/\mathbf{Z}; \mathbf{C})$.

- (i) The convolution $f * g$ is continuous and $\mathbf{Z}$-periodic. In other words, $f * g \in C(\mathbf{R}/\mathbf{Z}; \mathbf{C})$.

- (ii) We have $f * g = g * f$.

- (iii) We have $f * (g + h) = f * g + f * h$ and $(f + g) * h = f * h + g * h$. For any complex number c, we have $c(f * g) = (cf) * g = f * (cg)$.

- **Proof.** See Week 6 homework. $\square$

- Now we observe an interesting identity: for any $f \in C(\mathbf{R}/\mathbf{Z}; \mathbf{C})$ and any integer n, we have

$$f * e_n = \hat{f}(n)e_n.$$

- To prove this, we compute

$$f * e_n(x) = \int_{[0,1]} f(y) e^{2\pi i n(x-y)}\ dy$$

$$= e^{2\pi i n x} \int_{[0,1]} f(y) e^{-2\pi i n y}\ dy = \hat{f}(n) e^{2\pi i n x} = \hat{f}(n) e_n$$

as desired.

- More generally, we see from Lemma 6(iii) that for any trigonometric polynomial $P = \sum_{n=-N}^{n=N} c_n e_n$, we have

$$f * P = \sum_{n=-N}^{n=N} c_n (f * e_n) = \sum_{n=-N}^{n=N} \hat{f}(n) c_n e_n.$$

- In particular, the periodic convolution of any function in $C(\mathbf{R}/\mathbf{Z}; \mathbf{C})$ with a trigonometric polynomial, is again a trigonometric polynomial. (Compare with Lemma 6 of Week 4/5 notes).

- An optional remark: as a consequence of the above identity and Corollary 5, we have

$$\widehat{f * P}(n) = \hat{f}(n) c_n = \hat{f}(n) \hat{P}(n).$$

This is in fact part of a more general inequality, that

$$\widehat{f * g}(n) = \hat{f}(n) \hat{g}(n);$$

a fancy way of saying this is that the Fourier transform intertwines convolution and multiplication. While this identity is important (and is very useful in clarifying the nature of convolutions), we will not use it in this course.

- Next, we introduce the periodic analogue of an approximation to the identity.

- **Definition.** Let $\varepsilon > 0$ and $0 < \delta < 1/2$. A function $f \in C(\mathbf{R}/\mathbf{Z}; \mathbf{C})$ is said to be a *periodic* (ε, δ) *approximation to the identity* if the following properties are true:

- (i) $f(x) \geq 0$ for all $x \in \mathbf{R}$, and $\int_{[0,1]} f = 1$.
- (ii) We have $f(x) < \varepsilon$ for all $\delta \leq |x| \leq 1 - \delta$.
- Now we have the analogue of Lemma 4 from Week 4/5 notes:
- **Lemma 7** For every $\varepsilon > 0$ and $0 < \delta < 1/2$, there exists a trigonometric polynomial P which is an (ε, δ) approximation to the identity.
- **Proof.** We sketch the proof of this lemma here, and leave the completion of it as homework.
- Let $N \geq 1$. We define the *Fejér kernel* F_N to be the function

$$F_N = \sum_{n=-N}^{N} (1 - \frac{|n|}{N}) e_n.$$

Clearly F_N is a trigonometric polynomial. We observe the identity

$$F_N = \frac{1}{N} |\sum_{n=0}^{N-1} e_n|^2$$

(why?). But from the geometric series formula we have

$$\sum_{n=0}^{N-1} e_n(x) = \frac{e_N - e_0}{e_1 - e_0} = \frac{e^{\pi i (N-1)x} \sin(\pi N x)}{\sin(\pi x)}$$

when x is not an integer, (why?) and hence we have the formula

$$F_N(x) = \frac{\sin(\pi N x)^2}{N \sin(\pi x)^2}.$$

When x is an integer, the geometric series formula does not apply, but one has $F_N(x) = N$ in that case, as one can see by direct computation. In either case we see that $F_N(x) \geq 0$ for any x. Also, we have

$$\int_{[0,1]} F_N(x)\, dx = \sum_{n=-N}^{N} (1 - \frac{|n|}{N}) \int_{[0,1]} e_n = (1 - \frac{|0|}{N}) 1 = 1$$

(why?). Finally, since $\sin(\pi N x) \leq 1$, we have

$$F_N(x) \leq \frac{1}{N \sin(\pi x)^2} \leq \frac{1}{N \sin(\pi \delta)^2}$$

whenever $\delta < |x| < 1 - \delta$ (this is because sin is increasing on $[0, \pi/2]$ and decreasing on $[\pi/2, \pi]$). Thus by choosing N large enough, we can make $F_N(x) \leq \varepsilon$ for all $\delta < |x| < 1 - \delta$. $\square$

- Now we can prove the periodic Weierstrass approximation theorem. Let f be any element of $C(\mathbf{R}/\mathbf{Z}; \mathbf{C})$; we know that f is bounded, so that we have some $M > 0$ such that $|f(x)| \leq M$ for all $x \in \mathbf{R}$.

- Let $\varepsilon > 0$ be arbitrary. Since f is uniformly continuous, there exists a $\delta > 0$ such that $|f(x) - f(y)| \leq \varepsilon$ whenever $|x - y| \leq \delta$. Now use Lemma 7 to find a trigonometric polynomial P which is a (ε, δ) approximation to the identity. Then $f * P$ is also a trigonometric polynomial. We estimate $\|f - f * P\|_\infty$.

- Let x be any real number. We have

$$\begin{aligned}
|f(x) - f * P(x)| &= |f(x) - P * f(x)| \\
&= |f(x) - \textstyle\int_{[0,1]} f(x-y) P(y)\, dy| \\
&= |\textstyle\int_{[0,1]} f(x) P(y)\, dy - \int_{[0,1]} f(x-y) P(y)\, dy| \\
&= |\textstyle\int_{[0,1]} (f(x) - f(x-y)) P(y)\, dy| \\
&\leq \textstyle\int_{[0,1]} |f(x) - f(x-y)| P(y)\, dy \\
&\leq \textstyle\int_{[0,\delta]} |f(x) - f(x-y)| P(y)\, dy + \int_{[\delta, 1-\delta]} |f(x) - f(x-y)| P(y)\, dy \\
&\quad + \textstyle\int_{[1-\delta, 1]} |f(x) - f(x-y)| P(y)\, dy \\
&\leq \textstyle\int_{[0,\delta]} \varepsilon P(y)\, dy + \int_{[\delta, 1-\delta]} 2M\varepsilon\, dy \\
&\quad + \textstyle\int_{[1-\delta, 1]} |f(x-1) - f(x-y)| P(y)\, dy \\
&\leq \textstyle\int_{[0,\delta]} \varepsilon P(y)\, dy + \int_{[\delta, 1-\delta]} 2M\varepsilon\, dy \\
&\quad + \textstyle\int_{[1-\delta, 1]} \varepsilon P(y)\, dy \\
&\leq \varepsilon + 2M\varepsilon + \varepsilon \\
&= (2M + 2)\varepsilon.
\end{aligned}$$

Thus we have $\|f - f * P\|_\infty \leq (2M + 2)\varepsilon$. Since M is fixed and ε is arbitrary, we can thus make $f * P$ arbitrarily close to f in sup norm, which proves the periodic Weierstrass approximation theorem.

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The Fourier and Plancherel theorems

- Based on the Weierstrass approximation theorem, we can now generalize the Fourier and Plancherel identities to arbitrary continuous functions.

- **Fourier theorem** For any $f \in C(\mathbf{R}/\mathbf{Z}; \mathbf{C})$, the series $\sum_{n=-\infty}^{\infty} \hat{f}(n) e_n$ converges in L^2 metric to f. In other words, we have

$$\lim_{N \to \infty} \|f - \sum_{n=-N}^{N} \hat{f}(n) e_n\|_2 = 0.$$

- **Proof.** Let $\varepsilon > 0$. We have to show that there exists an N_0 such that $\|f - \sum_{n=-N}^{N} \hat{f}(n) e_n\|_2 \leq \varepsilon$ for all sufficiently large N.

- By the Weierstrass approximation theorem, we can find a trigonometric polynomial $P = \sum_{n=-N_0}^{N_0} c_n e_n$ such that $\|f - P\|_\infty \leq \varepsilon$, for some $N_0 > 0$. In particular we have $\|f - P\|_2 \leq \varepsilon$.

- Now let $N > N_0$, and let $F_N := \sum_{n=-N}^{n=N} \hat{f}(n) e_n$. We claim that $\|f - F_N\|_2 \leq \varepsilon$. First observe that for any $|m| \leq N$, we have

$$\langle f - F_N, e_m \rangle = \langle f, e_m \rangle - \sum_{n=-N}^{N} \hat{f}(n) \langle e_n, e_m \rangle = \hat{f}(m) - \hat{f}(m) = 0,$$

  where we have used Lemma 4. In particular we have

$$\langle f - F_N, F_N - P \rangle = 0$$

  since we can write $F_N - P$ as a linear combination of the e_m for which $|m| \leq N$. By Pythagoras's theorem we therefore have

$$\|f - P\|_2^2 = \|f - F_N\|_2^2 + \|F_N - P\|_2^2$$

  and in particular

$$\|f - F_N\|_2 \leq \|f - P\|_2 \leq \varepsilon$$

  as desired. $\square$

- Note that we have only obtained convergence of the Fourier series $\sum_{n=-\infty}^{\infty} \hat{f}(n)e_n$ to f in the L^2 metric. One may ask whether one has convergence in the uniform or pointwise sense as well, but it turns out (perhaps somewhat surprisingly) that the answer is no to both of those questions. However, if one assumes that the function f is not only continuous, but is also continuously differentiable, then one can recover pointwise convergence; if one assumes continuously twice differentiable, then one gets uniform convergence as well. We will not prove these results in this course, as they are a little tricky; they will be covered however in Math 133. However, we will prove one theorem about when one can improve the L^2 convergence to uniform convergence:

- **Theorem 8** Let $f \in C(\mathbf{R}/\mathbf{Z};\mathbf{C})$, and suppose that the series $\sum_{n=-\infty}^{\infty} |\hat{f}(n)|$ is absolutely convergent. Then the series $\sum_{n=-\infty}^{\infty} \hat{f}(n)e_n$ converges uniformly to f. In other words, we have

$$\lim_{N\to\infty} \|f - \sum_{n=-N}^{N} \hat{f}(n)e_n\|_\infty = 0.$$

- **Proof.** By the Weierstrass M-test, we see that $\sum_{n=-\infty}^{\infty} \hat{f}(n)e_n$ converges to SOME function F, which by Lemma 1(iii) is also continuous and $\mathbf{Z}$-periodic. (Strictly speaking, the Weierstrass M test was phrased for series from $n = 1$ to $n = \infty$, but also works for series from $n = -\infty$ to $n = +\infty$; this can be seen by splitting the doubly infinite series into two pieces). Thus

$$\lim_{N\to\infty} \|F - \sum_{n=-N}^{N} \hat{f}(n)e_n\|_\infty = 0$$

which implies that

$$\lim_{N\to\infty} \|F - \sum_{n=-N}^{N} \hat{f}(n)e_n\|_2 = 0$$

since the L^2 norm is always less than or equal to the L^∞ norm. But the sequence $\sum_{n=-N}^{N} \hat{f}(n)e_n$ is already converging in L^2 metric to f by the Fourier theorem, so can only converge in L^2 metric to F if $F = f$ (cf. Lemma 4 of Week 1 notes). Thus $F = f$, and so we have

$$\lim_{N\to\infty} \|f - \sum_{n=-N}^{N} \hat{f}(n)e_n\|_\infty = 0$$

as desired. $\square$

- As a corollary of the Fourier theorem, we can obtain the Plancherel identity:

- **Plancherel theorem** For any $f \in C(\mathbf{R}/\mathbf{Z}; \mathbf{C})$, the series $\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2$ is absolutely convergent, and

$$\|f\|_2^2 = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2.$$

- **Proof.** Let $\varepsilon > 0$. By the Fourier theorem we know that

$$\|f - \sum_{n=-N}^{N} \hat{f}(n)e_n\|_2 \leq \varepsilon$$

if N is large enough (depending on ε). In particular, by the triangle inequality this implies that

$$\|f\|_2 - \varepsilon \leq \| \sum_{n=-N}^{N} \hat{f}(n)e_n\|_2 \leq \|f\|_2 + \varepsilon.$$

On the other hand, by Corollary 5 we have

$$\| \sum_{n=-N}^{N} \hat{f}(n)e_n\|_2 = (\sum_{n=-N}^{N} |\hat{f}(n)|^2)^{1/2}$$

and hence

$$(\|f\|_2 - \varepsilon)^2 \leq \sum_{n=-N}^{N} |\hat{f}(n)|^2 \leq (\|f\|_2 + \varepsilon)^2.$$

Taking lim sup, we obtain

$$(\|f\|_2 - \varepsilon)^2 \leq \limsup_{N \to \infty} \sum_{n=-N}^{N} |\hat{f}(n)|^2 \leq (\|f\|_2 + \varepsilon)^2.$$

Since ε is arbitrary, we thus obtain by the squeeze test that

$$\limsup_{N \to \infty} \sum_{n=-N}^{N} |\hat{f}(n)|^2 = \|f\|_2^2$$

and the claim follows. □

- There are many other properties of the Fourier transform, but we will not develop them here. In the homework you will see a small application of the Fourier and Plancherel theorems.