Math 131BH - Weeks 4-5
Textbook pages covered: 152-154; 159-165; 172-184

- Uniform convergence and derivatives
- Uniform approximation by polynomials
- Power series
- Special functions

* * * * *

Uniform convergence and derivatives

- We have already seen in last week's notes how uniform convergence interacts well with continuity, with limits, and with integrals. Now we investigate how it interacts with derivatives.

- The first question we can ask is: if f_n converges uniformly to f, and the functions f_n are differentiable, does this imply that f is also differentiable? And does f_n' also converge to f'?

- The answer to the second question is, unfortunately, no. To see a counterexample, we will use without proof some basic facts about trigonometric functions (which we will make rigorous at the end of these notes). Consider the functions $f_n : [0, 2\pi] \to \mathbf{R}$ defined by $f_n(x) := n^{-1/2} \sin(nx)$, and let $f : [0, 2\pi] \to \mathbf{R}$ be the zero function $f(x) := 0$. Then, since sin takes values between -1 and 1, we have $d_\infty(f_n, f) \leq n^{-1/2}$, where we use the uniform metric $d_\infty(f, g) := \sup_{x \in [0,2\pi]} |f(x) - g(x)|$ introduced in the previous week's notes. Since $n^{-1/2}$ converges to 0, we thus see by the squeeze test that f_n converges uniformly to f. On the other hand, $f_n'(x) = n^{1/2} \cos(nx)$, and so in particular $|f_n'(0) - f'(0)| = n^{1/2}$. Thus f_n' does not converge pointwise to f', and so in particular does not converge uniformly either. In particular we have

$$\frac{d}{dx} \lim_{n\to\infty} f_n(x) \neq \lim_{n\to\infty} \frac{d}{dx} f_n(x).$$

- The answer to the first question is also no. An example is the sequence of functions $f_n : [-1,1] \to \mathbf{R}$ defined by $f_n(x) := \sqrt{\frac{1}{n^2} + x^2}$. These functions are differentiable (why?). Also, one can easily check that

$$|x| \leq f_n(x) \leq |x| + \frac{1}{n}$$

for all $x \in [-1,1]$ (why? square both sides), and so by the squeeze test f_n converges uniformly to the absolute value function $f(x) := |x|$. But this function is not differentiable at 0. Thus, the uniform limit of differentiable functions need not be differentiable.

- So, in summary, uniform convergence of the functions f_n says nothing about the convergence of the derivatives f_n'. However, the converse is true, as long as f_n converges at at least one point:

- **Theorem 1.** Let $[a,b]$ be an interval, and for every integer $n \geq 1$, let $f_n : [a,b] \to \mathbf{R}$ be a differentiable function whose derivative $f_n' : [a,b] \to \mathbf{R}$ is continuous. Suppose that the derivatives f_n' converge uniformly to a function $g : [a,b] \to \mathbf{R}$. Suppose also that there exists a point x_0 such that the limit $\lim_{n\to\infty} f_n(x_0)$ exists. Then the functions f_n converge uniformly to a differentiable function f, and the derivative of f equals g.

- Informally, the above theorem says that if f_n' converges uniformly, and $f_n(x_0)$ converges for some x_0, then f_n also converges uniformly, and $\frac{d}{dx} \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{d}{dx} f_n(x)$.

- **Proof.** We only give the beginning of the proof here; the remainder of the proof is given as a homework assignment.

- Since f_n' is continuous, we see from the fundamental theorem of calculus (see my Week 10 Math 131AH notes, or Theorem 6.21 of Rudin) that

$$f_n(x) - f_n(x_0) = \int_{[x_0,x]} f_n'$$

when $x \in [x_0, b]$, and

$$f_n(x) - f_n(x_0) = -\int_{[x,x_0]} f_n'$$

when $x \in [a, x_0]$.

- Let L be the limit of $f_n(x_0)$ as $n \to \infty$:
$$L := \lim_{n\to\infty} f_n(x_0).$$
By hypothesis, L exists. Now, since each f_n' is continuous on $[a, b]$, and f_n' converges uniformly to g, we see by Corollary 3 from last week's notes that g is also continuous. Now define the function $f : [a, b] \to \mathbf{R}$ by setting
$$f(x) := L - \int_{[a,x_0]} g + \int_{[a,x]} g$$
for all $x \in [a, b]$. To finish the proof, we have to show that f_n converges uniformly to f, and that f is differentiable with derivative g. (The rest of the proof is assigned as homework). $\square$

- It turns out that Theorem 1 is still true when the functions f_n' are not assumed to be continuous, but the proof is more difficult; see Rudin.

- By combining this theorem with the Weierstrass M-test, we obtain

- **Corollary 2.** Let $[a, b]$ be an interval, and for every integer $n \geq 1$, let $f_n : [a, b] \to \mathbf{R}$ be a differentiable function whose derivative $f_n' : [a, b] \to \mathbf{R}$ is continuous. Suppose that the series $\sum_{n=1}^\infty \|f_n'\|_\infty$ is absolutely convergent, where $\|f_n'\|_\infty := \sup_{x\in[a,b]} |f_n'(x)|$ is the sup norm of f_n', as described in last week's notes. Suppose also that the series $\sum_{n=1}^\infty f_n(x_0)$ is convergent for some $x_0 \in [a, b]$. Then the series $\sum_{n=1}^\infty f_n$ converges uniformly on $[a, b]$ to a differentiable function, and in fact
$$\frac{d}{dx}\sum_{n=1}^\infty f_n(x) = \sum_{n=1}^\infty \frac{d}{dx} f_n(x)$$
for all $x \in [a, b]$.

- **Proof.** See Week 4 homework. $\square$

- We now pause to give an example of a function which is continuous everywhere, but differentiable nowhere (this particular example was discovered by Weierstrass). Again, we will presume knowledge of the trigonometric functions.

- **Example.** Let $f : \mathbf{R} \to \mathbf{R}$ be the function $f(x) := \sum_{n=1}^\infty 4^{-n} \cos(32^n \pi x)$. Note that this series is uniformly convergent, thanks to the Weierstrass M-test, and since each individual function $4^{-n}\cos(32^n\pi x)$ is continuous, the function f is also continuous. However, it is not differentiable (see homework).

- A related, but slightly different, example of a continuous, nowhere differentiable function, appears in Rudin.

* * * * *

Uniform approximation by polynomials.

- As we have just seen, continuous functions can be very badly behaved, for instance they can be nowhere differentiable. On the other hand, functions such as polynomials are always very well behaved, in particular being always differentiable. Fortunately, while most continuous functions are not as well behaved as polynomials, they can always be *uniformly approximated* by polynomials; this important (but difficult) result is known as the *Weierstrass approximation theorem*, and is the subject of this section.

- **Definition.** Let $[a, b]$ be an interval. A *polynomial on* $[a, b]$ is a function $f : [a, b] \to \mathbf{R}$ of the form $f(x) := \sum_{j=0}^n c_j x^j$, where $n \geq 0$ is an integer and $c_0, \ldots, c_n$ are real numbers. If $c_n \neq 0$, then n is called the *degree* of f.

- **Example.** The function $f : [1, 2] \to \mathbf{R}$ defined by $f(x) := 3x^4 + 2x^3 - 4x + 5$ is a polynomial on $[1, 2]$ of degree 4.

- **Weierstrass approximation theorem.** If $[a, b]$ is an interval, $f : [a, b] \to \mathbf{R}$ is a continuous function, and $\varepsilon > 0$, then there exists a polynomial P on $[a, b]$ such that $d_\infty(P, f) \leq \varepsilon$ (i.e. $|P(x) - f(x)| \leq \varepsilon$ for all $x \in [a, b]$).

- Another way of stating this theorem is as follows. Recall that $C([a, b]; \mathbf{R})$ was the space of continuous functions from $[a, b]$ to $\mathbf{R}$, with the uniform metric d_∞. Let $P([a, b]; \mathbf{R})$ be the space of all polynomials on $[a, b]$; this is a subspace of $C([a, b]; \mathbf{R})$, since all polynomials are continuous. The Weierstrass approximation theorem then asserts that every continuous function is an adherent point of $P([a,b];\mathbf{R})$; or in other words, that the closure of the space of polynomials is the space of continuous functions:

$$\overline{P([a,b];\mathbf{R})} = C([a,b];\mathbf{R}).$$

In particular, every continuous function on $[a,b]$ is the uniform limit of polynomials.

- The proof of this theorem is somewhat complicated and will be done in stages. We first need the notion of an *approximation to the identity.*

- **Definition.** Let $[a,b]$ be an interval. A function $f : \mathbf{R} \to \mathbf{R}$ is said to be *supported* on $[a,b]$ iff $f(x) = 0$ for all $x \notin [a,b]$. We say that f is *compactly supported* iff it is supported on some interval $[a,b]$. If f is continuous and supported on $[a,b]$, we define the improper integral $\int_{-\infty}^{\infty} f$ to be $\int_{-\infty}^{\infty} f := \int_{[a,b]} f$.

- Note that a function can be supported on more than one interval, for instance a function which is supported on $[3,4]$ is also automatically supported on $[2,5]$ (why?). In principle, this might mean that our definition of $\int_{-\infty}^{\infty} f$ is not well defined, however this is not the case:

- **Lemma 3** If $f : \mathbf{R} \to \mathbf{R}$ is continuous and supported on an interval $[a,b]$, and is also supported on another interval $[c,d]$, then $\int_{[a,b]} f = \int_{[c,d]} f$.

- **Proof.** See Week 4 homework. $\square$

- **Definition** Let $\varepsilon > 0$ and $0 < \delta < 1$. A function $f : \mathbf{R} \to \mathbf{R}$ is said to be an (ε,δ)-*approximation to the identity* if it obeys the following three properties:

- (a) f is supported on $[-1,1]$, and $f(x) \geq 0$ for all $-1 \leq x \leq 1$.

- (b) f is continuous, and $\int_{-\infty}^{\infty} f = 1$.

- (c) $|f(x)| \leq \varepsilon$ for all $\delta \leq |x| \leq 1$.

- Optional remark: For those of you who are familiar with the Dirac delta function, approximations to the identity are ways to approximate this (very discontinuous) delta function by a continuous function (which is easier to analyze).

- Our proof of the Weierstrass approximation theorem relies on three key facts. The first fact is that polynomials can be approximations to the identity:

- **Lemma 4.** For every $\varepsilon > 0$ and every $0 < \delta < 1$ there exists an (ε, δ)-approximation to the identity which is a polynomial P on $[-1, 1]$.

- **Proof.** See Week 4 homework. $\square$

- We will use these polynomial approximations to the identity to approximate continuous functions by polynomials. We will need the following important notion, however, that of a *convolution*.

- **Definition.** Let $f : \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$ be continuous, compactly supported functions. We define the *convolution* $f * g : \mathbf{R} \to \mathbf{R}$ of f and g to be the function

$$(f * g)(x) := \int_{-\infty}^{\infty} f(y)g(x - y)\ dy.$$

- Note that if f and g are continuous and compactly supported, then for each x the function $f(y)g(x - y)$ (thought of as a function of y) is also continuous and compactly supported, so the above definition makes sense.

- Convolutions play an important role in Fourier analysis and in PDE, and are also important in physics, engineering, and signal processing. An in-depth study of convolution would occupy an entire course; for now, we only give a very brief discussion. We begin by listing some basic properties of convolution.

- **Proposition 5.** Let $f : \mathbf{R} \to \mathbf{R}$, $g : \mathbf{R} \to \mathbf{R}$, and $h : \mathbf{R} \to \mathbf{R}$ be continuous, compactly supported functions. Then the following statements are true.

- (a) The convolution $f * g$ is also a continuous, compactly supported function.

- (b) (Convolution is commutative) We have $f * g = g * f$; in other words

$$f * g(x) = \int_{-\infty}^{\infty} f(y)g(x-y)\ dy = \int_{-\infty}^{\infty} g(y)f(x-y)\ dy = g * f(x).$$

- (c) (Convolution is linear) We have $f * (g+h) = f * g + f * h$. Also, for any real number c, we have $f * (cg) = (cf) * g = c(f * g)$.

- **Proof.** See Week 4 homework. $\square$

- Optional remark: There are many other important properties of convolution, for instance it is associative, $(f * g) * h = f * (g * h)$, and it commutes with derivatives, $(f * g)' = f' * g = f * g'$, when f and g are differentiable. The Dirac delta function δ mentioned earlier is an identity for convolution: $f * \delta = \delta * f = f$. These results are slightly harder to prove than the ones in Proposition 5, however, and we will not need them in this course.

- As mentioned earlier, the proof of the Weierstrass approximation theorem relies on three facts. The second key fact is that convolution with polynomials produces another polynomial:

- **Lemma 6.** Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous function supported on $[0,1]$, and let $g : \mathbf{R} \to \mathbf{R}$ be a continuous function supported on $[-1,1]$ which is a polynomial on $[-1,1]$. Then $f * g$ is a polynomial on $[0,1]$. (Note however that it may be non-polynomial outside of $[0,1]$).

- **Proof.** Since g is polynomial on $[-1,1]$, we may find an integer $n \geq 0$ and real numbers $c_0, c_1, \ldots, c_n$ such that

$$g(x) = \sum_{j=0}^{n} c_j x^j \text{ for all } x \in [-1,1].$$

On the other hand, for all $x \in [0,1]$, we have

$$f * g(x) = \int_{-\infty}^{\infty} f(y)g(x-y)\ dy = \int_{[0,1]} f(y)g(x-y)\ dy$$

since f is supported on $[0,1]$. Since $x \in [0,1]$ and the variable of integration y is also in $[0,1]$, we have $x - y \in [-1,1]$. Thus we may substitute in our formula for g to obtain

$$f * g(x) = \int_{[0,1]} f(y) \sum_{j=0}^{n} c_j (x-y)^j \, dy.$$

We expand this using the binomial formula to obtain

$$f * g(x) = \int_{[0,1]} f(y) \sum_{j=0}^{n} c_j \sum_{k=0}^{j} \frac{j!}{k!(j-k)!} x^k (-y)^{j-k} \, dy.$$

We can interchange the two summations (this is OK since both sums are finite) to obtain

$$f * g(x) = \int_{[0,1]} f(y) \sum_{k=0}^{n} \sum_{j=k}^{n} c_j \frac{j!}{k!(j-k)!} x^k (-y)^{j-k} \, dy$$

(why did the limits of integration change? It may help to plot j and k on a graph). Now we interchange the k summation with the integral, and observe that x^k is independent of y, to obtain

$$f * g(x) = \sum_{k=0}^{n} x^k \int_{[0,1]} f(y) \sum_{j=k}^{n} c_j \frac{j!}{k!(j-k)!} (-y)^{j-k} \, dy.$$

If we thus define

$$C_k := \int_{[0,1]} f(y) \sum_{j=k}^{n} c_j \frac{j!}{k!(j-k)!} (-y)^{j-k} \, dy$$

for each $k = 0, \ldots, n$, then C_k is a number which is independent of x, and we have

$$f * g(x) = \sum_{k=0}^{n} C_k x^k$$

for all $x \in [0,1]$. Thus $f * g$ is a polynomial on $[0,1]$. $\square$

- The third key fact is that if one convolves a uniformly continuous function with an approximation to the identity, we obtain a new function which is close to the original function:

- **Lemma 7.** Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous function supported on $[0,1]$, which is bounded by some $M > 0$ (i.e. $|f(x)| \leq M$ for all $x \in \mathbf{R}$), and let $\varepsilon > 0$ and $0 < \delta < 1$ be such that one has $|f(x) - f(y)| < \varepsilon$ whenever $x, y \in \mathbf{R}$ and $|x - y| < \delta$. Let g be any (ε, δ)-approximation to the identity. Then we have

$$|f * g(x) - f(x)| \leq (3M + 2\delta)\varepsilon$$

  for all $x \in [0,1]$.

- **Proof.** See Week 4 homework. $\square$

- Combining these together, we obtain a preliminary version of the Weierstrass approximation theorem:

- **Corollary 8.** Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous function supported on $[0,1]$. Then for every $\varepsilon > 0$, there exists a function $P : \mathbf{R} \to \mathbf{R}$ which is polynomial on $[0,1]$ and such that $|P(x) - f(x)| \leq \varepsilon$ for all $x \in [0,1]$.

- **Proof.** See Week 4 homework. $\square$

- Now we perform a series of modifications to convert Corollary 8 into the actual Weierstrass approximation theorem. We first need a simple lemma.

- **Lemma 9.** Let $f : [0,1] \to \mathbf{R}$ be a continuous function which equals 0 on the boundary of $[0,1]$, i.e. $f(0) = f(1) = 0$. Let $F : \mathbf{R} \to \mathbf{R}$ be the function defined by setting $F(x) := f(x)$ for $x \in [0,1]$ and $F(x) := 0$ for $x \notin [0,1]$. Then F is also continuous.

- **Proof.** See Week 4 homework. $\square$

- The function F obtained in Lemma 9 is sometimes known as the *extension of f by zero*.

- From Corollary 8 and Lemma 9 we immediately obtain

- **Corollary 10.** Let $f : [0,1] \to \mathbf{R}$ be a continuous function supported on $[0,1]$ such that $f(0) = f(1) = 0$. Then for every $\varepsilon > 0$ there exists a polynomial $P : [0,1] \to \mathbf{R}$ such that $|P(x) - f(x)| \le \varepsilon$ for all $x \in [0,1]$.

- Now we strengthen Corollary 10 by removing the assumption that $f(0) = f(1) = 0$.

- **Corollary 11.** Let $f : [0,1] \to \mathbf{R}$ be a continuous function supported on $[0,1]$. Then for every $\varepsilon > 0$ there exists a polynomial $P : [0,1] \to \mathbf{R}$ such that $|P(x) - f(x)| \le \varepsilon$ for all $x \in [0,1]$.

- **Proof.** Let $F : [0,1] \to \mathbf{R}$ denote the function

  $$F(x) := f(x) - f(0) - x(f(1) - f(0)).$$

  Observe that F is also continuous (why?), and that $F(0) = F(1) = 0$. By Corollary 10, we can thus find a polynomial $Q : [0,1] \to \mathbf{R}$ such that $|Q(x) - F(x)| \le \varepsilon$ for all $x \in [0,1]$. But

  $$Q(x) - F(x) = Q(x) + f(0) + x(f(1) - f(0)) - f(x),$$

  so the claim follows by setting P to be the polynomial $P(x) := Q(x) + f(0) + x(f(1) - f(0))$. $\square$

- Finally, we can prove the Weierstrass approximation theorem.

- **Proof of Weierstrass approximation theorem.** Let $f : [a,b] \to \mathbf{R}$ be a continuous function on $[a,b]$. Let $g : [0,1] \to \mathbf{R}$ denote the function

  $$g(x) := f(a + (b-a)x) \text{ for all } x \in [0,1]$$

  Observe then that

  $$f(y) = g((y-a)/(b-a)) \text{ for all } y \in [a,b].$$

  The function g is continuous on $[0,1]$ (why?), and so by Corollary 11 we may find a polynomial $Q : [0,1] \to \mathbf{R}$ such that $|Q(x) - g(x)| \le \varepsilon$ for all $x \in [0,1]$. In particular, for any $y \in [a,b]$, we have

  $$|Q((y-a)/(b-a)) - g((y-a)/(b-a))| \le \varepsilon.$$

  If we thus set $P(y) := Q((y-a)/(b-a))$, then we observe that P is also a polynomial (why?), and so we have $|P(y) - g(y)| \le \varepsilon$ for all $y \in [a,b]$, as desired. $\square$

- Note that the Weierstrass approximation theorem only works on bounded intervals $[a, b]$; continuous functions on $\mathbf{R}$ cannot be uniformly approximated by polynomials. For instance, the exponential function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) := e^x$ cannot be approximated by any polynomial, because exponential functions grow faster than any polynomial and so there is no way one can even make the sup metric between f and a polynomial finite.

- There is a generalization of the Weierstrass approximation theorem to higher dimensions: if K is any compact subset of $\mathbf{R}^n$ (with the Euclidean metric d_{l^2}), and $f : K \to \mathbf{R}$ is a continuous function, then for every $\varepsilon > 0$ there exists a polynomial $P : K \to \mathbf{R}$ of n variables $x_1, \ldots, x_n$ such that $d_\infty(f, P) < \varepsilon$. The proof of this more general theorem is in Rudin, however we will not need it here.

* * * * *

Power series

- We now discuss an important subclass of series of functions, that of *power series.*

- **Definition.** Let a be a real number. A *formal power series centered at a* is any series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

where $c_0, c_1, \ldots$ is a sequence of real numbers (not depending on x); we refer to c_n as the n^{th} *coefficient* of this series. Note that each term $c_n(x-a)^n$ in this series is a function of a real variable x.

- **Example** The series $\sum_{n=0}^\infty n!(x-2)^n$ is a formal power series centered at 2. The series $\sum_{n=0}^\infty 2^x (x-3)^n$ is not a formal power series, since the coefficients 2^x depend on x.

- We call these power series *formal* because we do not yet assume that these series converge for any x. However, these series are automatically guaranteed to converge when $x = a$ (why?). In general, the closer x gets to a, the easier it is for this series to converge. A more precise statement is as follows.

- **Definition.** Let $\sum_{n=0}^\infty c_n(x-a)^n$ be a formal power series. We define the *radius of convergence* R of this series to be the quantity

$$R := \frac{1}{\limsup_{n\to\infty} |c_n|^{1/n}}$$

where we adopt the convention that $\frac{1}{0} = +\infty$ and $\frac{1}{+\infty} = 0$.

- Note that each number $|c_n|^{1/n}$ is non-negative, so $\limsup_{n\to\infty} |c_n|^{1/n}$ can take on any value from 0 to $+\infty$, inclusive. Thus R can also take on any value between 0 and $+\infty$ inclusive (in particular it is not necessarily a real number). Note that the radius of convergence always exists, even if the sequence $|c_n|^{1/n}$ is not convergent, because the lim sup of any sequence always exists (though it might be $+\infty$ or $-\infty$).

- **Example.** The radius of convergence of the series $\sum_{n=0}^\infty n(-2)^n(x-3)^n$ is

$$\frac{1}{\limsup_{n\to\infty} |n(-2^n)|^{1/n}} = \frac{1}{\limsup_{n\to\infty} 2n^{1/n}} = \frac{1}{2}.$$

The radius of convergence of the series $\sum_{n=0}^\infty 2^{n^2}(x+2)^n$ is

$$\frac{1}{\limsup_{n\to\infty} |2^{n^2}|^{1/n}} = \frac{1}{\limsup_{n\to\infty} 2^n} = \frac{1}{+\infty} = 0.$$

The radius of convergence of the series $\sum_{n=0}^\infty 2^{-n^2}(x+2)^n$ is

$$\frac{1}{\limsup_{n\to\infty} |2^{-n^2}|^{1/n}} = \frac{1}{\limsup_{n\to\infty} 2^{-n}} = \frac{1}{0} = +\infty.$$

- The significance of the radius of convergence is the following.

- **Theorem 12.** Let $\sum_{n=0}^\infty c_n(x-a)^n$ be a formal power series, and let R be its radius of convergence.

- (a) (Divergence outside of the radius of convergence) If $x \in \mathbf{R}$ is such that $|x-a| > R$, then the series $\sum_{n=0}^\infty c_n(x-a)^n$ is divergent for that value of x.

- (b) (Convergence inside the radius of convergence) If $x \in \mathbf{R}$ is such that $|x-a| < R$, then the series $\sum_{n=0}^\infty c_n(x-a)^n$ is absolutely convergent for that value of x.

- For parts (cde) we assume that $R > 0$ (i.e. the series converges at at least one other point than $x = a$). Let $f : (a - R, a + R) \to \mathbf{R}$ be the function $f(x) := \sum_{n=0}^{\infty} c_n(x-a)^n$; this function is guaranteed to exist by (b).

- (c) (Uniform convergence on compacta) For any $0 < r < R$, the series $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges uniformly to f on the compact interval $[a - r, a + r]$. In particular, f is continuous on $(a - R, a + R)$.

- (d) (Differentiation of power series) The function f is differentiable on $(a - R, a + R)$, and for any $0 < r < R$, the series $\sum_{n=0}^{\infty} nc_n(x-a)^{n-1}$ converges uniformly to f' on the interval $[a - r, a + r]$.

- (e) (Integration of power series) For any closed interval $[y, z]$ contained in $(a - R, a + R)$, we have

$$\int_{[y,z]} f = \sum_{n=0}^{\infty} c_n \frac{(z-a)^{n+1} - (y-a)^{n+1}}{n+1}.$$

- **Proof.** See Week 5 homework. □

- Parts (a) and (b) of the above theorem give another way to find the radius of convergence, by using your favorite convergence test to work out the range of x for which the power series converges:

- **Example.** Consider the power series $\sum_{n=0}^{\infty} n(x-1)^n$. The ratio test shows that this series converges when $|x - 1| < 1$ and diverges when $|x - 1| > 1$ (why?). Thus the only possible value for the radius of convergence is $R = 1$ (if $R < 1$, then we have contradicted Theorem 12(a); if $R > 1$, then we have contradicted Theorem 12(b)).

- Note that Theorem 12 is silent on what happens when $|x - a| = R$, i.e. at the points $a - R$ and $a + R$. Indeed, one can have either convergence or divergence at those points; see homework.

- Notice in Theorem 12 that while a power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ will converge pointwise on the interval $(a - R, a + R)$, it need not converge uniformly on that interval (see homework). On the other hand, Theorem 12(c) assures us that the power series will converge on any smaller interval $[a-r, a+r]$. In particular, being uniformly convergent on every closed subinterval of $(a-R, a+R)$ is not enough to guarantee being uniformly convergent on all of $(a-R, a+R)$.

- A function $f(x)$ which is lucky enough to be representable as a power series have a special name; they are *real analytic.*

- **Definition** Let $(a-r, a+r)$ be an open interval, and let $f : E \to \mathbf{R}$ be a function defined on a set $E \subseteq \mathbf{R}$ which contains $(a-r, a+r)$. We say that f is *real analytic on* $(a-r, a+r)$ iff there exists a power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ centered at a which has a radius of convergence greater than or equal to r, and which converges to f on $(a-r, a+r)$.

- **Example.** The function $f : (0,2) \to \mathbf{R}$ defined by $f(x) = \sum_{n=0}^{\infty} n(x-1)^n$ is real analytic on $(0, 2)$.

- The notion of being real analytic is closely related to another notion, that of being *complex analytic*, but this is a topic for Math 132, Complex Analysis, and will not be discussed here.

- We now discuss which functions are real analytic. From Theorem 12(cd) we see that if f is real analytic on $(a-r, a+r)$, then f is both continuous and differentiable on $(a-r, a+r)$. We can in fact say more:

- **Definition** Let E be a subset of $\mathbf{R}$. We say a function $f : E \to \mathbf{R}$ is *once differentiable on* E iff it is differentiable. More generally, for any $k \geq 2$ we say that $f : E \to \mathbf{R}$ is k *times differentiable on* E, or just k *times differentiable*, iff f is differentiable, and f' is $k-1$ times differentiable. If f is k times differentiable, we define the k^{th} derivative $f^{(k)} : E \to \mathbf{R}$ by the recursive rule $f^{(1)} := f'$, and $f^{(k)} = (f^{(k-1)})'$ for all $k \geq 2$. We also define $f^{(0)} := f$ (this is f differentiated 0 times), and we allow every function to be zero times differentiable (since clearly $f^{(0)}$ exists for every f). A function is said to be *infinitely differentiable* iff it is k times differentiable for every $k \geq 0$.

- Thus for instance, the function $f(x) := |x|^3$ is twice differentiable on $\mathbf{R}$, but not three times differentiable (why?). Indeed, $f^{(2)} = f'' = 6|x|$, which is not differentiable, at 0.

- **Proposition 13.** Let f of a function real analytic on the interval $(a-r, a+r)$, with the power series expansion

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

for all $x \in (a-r, a+r)$. Then for any integer $k \geq 0$, the function f is k times differentiable on $(a-r, a+r)$, and the k^{th} derivative is given by

$$f^{(k)}(x) = \sum_{n=0}^{\infty} c_{n+k}(n+1)(n+2)\dots(n+k)(x-a)^n$$

for all $x \in (a-r, a+r)$.

- **Proof.** See Week 5 homework. □

- **Corollary 14 (Taylor's formula).** Let f of a function real analytic on the interval $(a-r, a+r)$, with the power series expansion

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

for all $x \in (a-r, a+r)$. Then for any integer $k \geq 0$, we have

$$f^{(k)}(a) = k! c_k,$$

where $k! := 1 \times 2 \times \dots \times k$ (and we adopt the convention that $0! = 1$). In particular, we have *Taylor's formula*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

for all x in $(a-r, a+r)$.

- **Proof.** See Week 5 homework. □

- The power series $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ is sometimes called the *Taylor series* of f around a. Taylor's formula thus asserts that if a function is analytic, then it is equal to its Taylor series.

- Note that Taylor's formula only works for functions which are analytic; there are examples of functions which are infinitely differentiable but for which Taylor's theorem fails (see Week 5 homework).

- Another corollary of Taylor's formula is that a real analytic function can have at most one power series at a point:

- **Corollary 15 (uniqueness of power series).** Let f of a function real analytic on the interval $(a-r, a+r)$, with two power series expansions

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

and

$$f(x) = \sum_{n=0}^{\infty} d_n (x-a)^n.$$

Then $c_n = d_n$ for all $n \geq 0$.

- **Proof.** By Corollary 14, we have $f^{(k)}(a) = k!c_k$ for all $k \geq 0$. But we also have $f^{(k)}(a) = k!d_k$, by similar reasoning. Since $k!$ is never zero, we can cancel it and obtain $c_k = d_k$ for all $k \geq 0$, as desired. $\square$

- On the other hand, an analytic function can certainly have different power series at different points. For instance, the function $f(x) := \frac{1}{1-x}$, defined on $\mathbf{R} - \{1\}$, has the power series

$$f(x) := \sum_{n=0}^{\infty} x^n$$

around 0, on the interval $(-1, 1)$, but also has the power series

$$f(x) = \frac{1}{1-x} = \frac{2}{1-2(x-\frac{1}{2})} = \sum_{n=0}^{\infty} 2(2(x-\frac{1}{2}))^n = \sum_{n=0}^{\infty} 2^{n+1}(x-\frac{1}{2})^n$$

around 1/2, on the interval $(0, 1)$ (note that the above power series has a radius of convergence of 1/2, thanks to the root test. Indeed, it can be proven that if f is analytic on any interval $(a-r, a+r)$, and $(b-s, b+s)$ is a sub-interval of $(a-r, a+r)$, then f is also analytic on

$(b-s, b+s)$; in particular, it has a power series expansion at b. However, we will not do so here, because this theorem is easiest to prove using the machinery of complex analysis and contour integration, as you will see in Math 132, Complex Analysis.

* * * * *

Abel's theorem

- Let $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ be a power series centered at a with a radius of convergence $0 < R < \infty$ strictly between 0 and infinity. From Theorem 12 we know that the power series converges absolutely whenever $|x-a| < R$, and diverges when $|x-a| > R$. However, at the boundary $|x-a| = R$ the situation is more complicated; the series may either converge or diverge (see homework). However, if the series does converge at the boundary point, then it is reasonably well behaved; in particular, it is continuous at that boundary point.

- **Abel's theorem.** Let $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ be a power series centered at a with radius of convergence $0 < R < \infty$. If the power series converges at $a+R$, then f is continuous at $a+R$, i.e.

$$\lim_{x \to a+R : x \in (a-R, a+R)} \sum_{n=0}^{\infty} c_n(x-a)^n = \sum_{n=0}^{\infty} c_n R^n.$$

  Similarly, if the power series converges at $a-R$, then f is continuous at $a-R$, i.e.

$$\lim_{x \to a-R : x \in (a-R, a+R)} \sum_{n=0}^{\infty} c_n(x-a)^n = \sum_{n=0}^{\infty} c_n(-R)^n.$$

- Before we prove Abel's theorem, we need the following lemma.

- **Summation by parts formula.** Let $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ be sequences of real numbers which converge to limits A and B respectively, i.e. $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$. Suppose that the sum $\sum_{n=0}^{\infty}(a_{n+1} - a_n)b_n$ is convergent. Then the sum $\sum_{n=0}^{\infty} a_{n+1}(b_{n+1} - b_n)$ is also convergent, and

$$\sum_{n=0}^{\infty}(a_{n+1} - a_n)b_n = AB - a_0 b_0 - \sum_{n=0}^{\infty} a_{n+1}(b_{n+1} - b_n).$$

- **Proof.** See Week 5 homework. $\square$

- Note that the summation by parts formula has a certain similarity with the more well-known *integration by parts formula*

$$\int_0^\infty f'(x)g(x)\ dx = f(x)g(x)|_0^\infty - \int_0^\infty f(x)g'(x)\ dx.$$

Indeed, the two are actually quite closely related.

- Now we can prove Abel's theorem. It will suffice to prove the first claim, i.e. that

$$\lim_{x\to a+R: x\in(a-R,a+R)} \sum_{n=0}^\infty c_n(x-a)^n = \sum_{n=0}^\infty c_n R^n$$

whenever the sum $\sum_{n=0}^\infty c_n R^n$ converges; the second claim will then follow (why?) by replacing c_n by $(-1)^n c_n$ in the above claim. If we make the substitutions $d_n := c_n R^n$ and $y := \frac{x-a}{R}$, then the above claim can be rewritten as

$$\lim_{y\to 1: y\in(-1,1)} \sum_{n=0}^\infty d_n y^n = \sum_{n=0}^\infty d_n$$

whenever the sum $\sum_{n=0}^\infty d_n$ converges. (Why is this equivalent to the previous claim?)

- Write $D := \sum_{n=0}^\infty d_n$, and for every $N \geq 0$ write

$$s_N := (\sum_{n=0}^{N-1} d_n) - D$$

so in particular $s_0 = -D$. Then observe that $\lim_{N\to\infty} S_N = 0$, and that $d_n = S_{n+1} - S_n$. Thus for any $y \in (-1,1)$ we have

$$\sum_{n=0}^\infty d_n y^n = \sum_{n=0}^\infty (S_{n+1} - S_n) y^n.$$

Applying the summation by parts formula, and noting that $\lim_{n\to\infty} y^n = 0$, we obtain

$$\sum_{n=0}^{\infty} d_n y^n = -S_0 y^0 - \sum_{n=0}^{\infty} S_{n+1}(y^{n+1} - y^n).$$

Observe that $-S_0 y^0 = +D$. Thus to finish the proof of Abel's theorem, it will suffice to show that

$$\lim_{y\to 1: y\in(-1,1)} \sum_{n=0}^{\infty} S_{n+1}(y^{n+1} - y^n) = 0.$$

Since y is converging to 1, we may as well restrict y to $[0,1)$ instead of $(-1,1)$; in particular we may take y to be positive.

- From the triangle inequality for series, we have

$$|\sum_{n=0}^{\infty} S_{n+1}(y^{n+1} - y^n)| \leq \sum_{n=0}^{\infty} |S_{n+1}(y^{n+1} - y^n)| = \sum_{n=0}^{\infty} |S_{n+1}|(y^n - y^{n+1}),$$

so by the squeeze test it suffices to show that

$$\lim_{y\to 1: y\in(-1,1)} \sum_{n=0}^{\infty} |S_{n+1}|(y^n - y^{n+1}) = 0.$$

Since the expression $\sum_{n=0}^{\infty} |S_{n+1}|(y^n - y^{n+1})$ is clearly non-negative, it will suffice to show that

$$\lim\sup_{y\to 1: y\in(-1,1)} \sum_{n=0}^{\infty} |S_{n+1}|(y^n - y^{n+1}) = 0.$$

Let $\varepsilon > 0$. Since S_n converges to 0, there exists an N such that $|S_n| \leq \varepsilon$ for all $n > N$. Thus we have

$$\sum_{n=0}^{\infty} |S_{n+1}|(y^n - y^{n+1}) \leq \sum_{n=0}^{N} |S_{n+1}|(y^n - y^{n+1}) + \sum_{n=N+1}^{\infty} \varepsilon(y^n - y^{n+1}).$$

The last summation is a telescoping series, which sums to εy^{N+1} (why? Note that $y^n \to 0$ as $n \to \infty$, and thus

$$\sum_{n=0}^{\infty} |S_{n+1}|(y^n - y^{n+1}) \leq \sum_{n=0}^{N} |S_{n+1}|(y^n - y^{n+1}) + \varepsilon y^{N+1}.$$

Now take limits as $y \to 1$. Observe that $y^n - y^{n+1} \to 0$ as $y \to 1$ for every $n \in 0, 1, \ldots, N$. Since we can interchange limits and *finite* sums, we thus have

$$\limsup_{n\to\infty} \sum_{n=0}^{\infty} |S_{n+1}|(y^n - y^{n+1}) \leq \varepsilon.$$

But $\varepsilon > 0$ was arbitrary, and thus we must have

$$\limsup_{n\to\infty} \sum_{n=0}^{\infty} |S_{n+1}|(y^n - y^{n+1}) = 0$$

since the left-hand side must be non-negative. The claim follows. $\square$

- We will see some applications of Abel's theorem later.

* * * * *

Multiplication of power series

- We now show that the product of two analytic functions is again analytic.

- **Theorem 16** Let $f : (a - r, a + r) \to \mathbf{R}$ and $g : (a - r, a + r) \to \mathbf{R}$ be functions analytic on $(a - r, a + r)$, with power series expansions

$$f(x) = \sum_{n=0}^{\infty} c_n (x - a)^n$$

and

$$g(x) = \sum_{n=0}^{\infty} d_n (x - a)^n$$

respectively. Then $fg : (a-r, a+r) \to \mathbf{R}$ is also analytic on $(a-r, a+r)$, with power series expansion

$$f(x)g(x) = \sum_{n=0}^\infty e_n(x-a)^n$$

where $e_n := \sum_{m=0}^n c_m d_{n-m}$.

- **Remark.** The sequence $(e_n)_{n=0}^\infty$ is sometimes referred to as the *convolution* of the sequences $(c_n)_{n=0}^\infty$ and $(d_n)_{n=0}^\infty$; it is closely related to the notion of convolution discussed earlier, in the proof of the Weierstrass approximation theorem.

- **Proof.** We have to show that the series $\sum_{n=0}^\infty e_n(x-a)^n$ converges to $f(x)g(x)$ for all $x \in (a-r, a+r)$. Now fix x to be any point in $(a-r, a+r)$. By Theorem 12, we see that both f and g have radii of convergence at least r. In particular, the series $\sum_{n=0}^\infty c_n(x-a)^n$ and $\sum_{n=0}^\infty d_n(x-a)^n$ are absolutely convergent. Thus if we define

$$C := \sum_{n=0}^\infty |c_n(x-a)^n|$$

and

$$D := \sum_{n=0}^\infty |d_n(x-a)^n|$$

then C and D are both finite.

- For any $N \geq 0$, consider the partial sum

$$\sum_{n=0}^N \sum_{m=0}^\infty |c_m(x-a)^m d_n(x-a)^n|.$$

We can rewrite this as

$$\sum_{n=0}^N |d_n(x-a)^n| \sum_{m=0}^\infty |c_m(x-a)^m|,$$

which by definition of C is equal to

$$\sum_{n=0}^N |d_n(x-a)^n| C,$$

which by definition of D is less than or equal to DC. Thus the above partial sums are bounded by DC for every N. In particular, the series

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}|c_m(x-a)^m d_n(x-a)^n|$$

is convergent, which means that the sum

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}c_m(x-a)^m d_n(x-a)^n$$

is absolutely convergent.

- Let us now compute this sum in two ways. First of all, we can pull the $d_n(x-a)^n$ factor out of the $\sum_{m=0}^{\infty}$ summation, to obtain

$$\sum_{n=0}^{\infty}d_n(x-a)^n\sum_{m=0}^{\infty}c_m(x-a)^m.$$

By our formula for $f(x)$, this is equal to

$$\sum_{n=0}^{\infty}d_n(x-a)^n f(x);$$

by our formula for $g(x)$, this is equal to $f(x)g(x)$. Thus

$$f(x)g(x)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}c_m(x-a)^m d_n(x-a)^n.$$

Now we compute this sum in a different way. We rewrite it as

$$f(x)g(x)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}c_m d_n(x-a)^{n+m}.$$

By Fubini's theorem for series (see Week 5 notes from my Math 131AH class), because the series was absolutely convergent, we may rewrite it as

$$f(x)g(x)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}c_m d_n(x-a)^{n+m}.$$

Now make the substitution $n' := n + m$, to rewrite this as

$$f(x)g(x) = \sum_{m=0}^{\infty} \sum_{n'=m}^{\infty} c_m d_{n'-m}(x-a)^{n'}.$$

If we adopt the convention that $d_j = 0$ for all negative j, then this is equal to

$$f(x)g(x) = \sum_{m=0}^{\infty} \sum_{n'=0}^{\infty} c_m d_{n'-m}(x-a)^{n'}.$$

Applying Fubini's theorem again, we obtain

$$f(x)g(x) = \sum_{n'=0}^{\infty} \sum_{m=0}^{\infty} c_m d_{n'-m}(x-a)^{n'},$$

which we can rewrite as

$$f(x)g(x) = \sum_{n'=0}^{\infty} (x-a)^{n'} \sum_{m=0}^{\infty} c_m d_{n'-m}.$$

Since d_j was 0 when j is negative, we can rewrite this as

$$f(x)g(x) = \sum_{n'=0}^{\infty} (x-a)^{n'} \sum_{m=0}^{n'} c_m d_{n'-m},$$

which by definition of e is

$$f(x)g(x) = \sum_{n'=0}^{\infty} e_{n'}(x-a)^{n'},$$

as desired. $\square$

* * * * *

The exponential and logarithm functions

- We can now use the machinery developed in the last few sections to develop a rigorous foundation for many standard functions used in mathematics. We begin with the exponential function.

- **Definition** For every real number x, we define the *exponential function* $\exp(x)$ to be the real number

$$\exp(x) := \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

- We now list the basic properties of the exponential function.

- **Theorem 17**

- (a) For every real number x, the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is absolutely convergent. In particular, $\exp(x)$ exists and is real for every $x \in \mathbf{R}$, the power series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ has an infinite radius of convergence, and exp is an analytic function on $(-\infty, \infty)$.

- (b) exp is differentiable on $\mathbf{R}$, and for every $x \in \mathbf{R}$, $\exp'(x) = \exp(x)$.

- (c) exp is continuous on $\mathbf{R}$, and for every interval $[a, b]$, we have $\int_{[a,b]} \exp(x)\, dx = \exp(b) - \exp(a)$.

- (d) For every $x, y \in \mathbf{R}$, we have $\exp(x + y) = \exp(x)\exp(y)$.

- (e) We have $\exp(0) = 1$. Also, for every $x \in \mathbf{R}$, $\exp(x)$ is positive, and $\exp(-x) = 1/\exp(x)$.

- (f) exp is strictly monotone increasing: in other words, if x, y are real numbers, then we have $\exp(y) > \exp(x)$ if and only if $y > x$.

- **Proof.** See Week 5 homework. $\square$

- **Definition** The number e is defined to be

$$e := \exp(1) = \sum_{n=0}^{\infty} \frac{1}{n!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots.$$

- **Proposition 18** For every real number x, we have $\exp(x) = e^x$.

- **Proof.** See Week 5 homework. $\square$

- We will thus use e^x and $\exp(x)$ interchangeably.

- Since $e > 1$ (why?), we see that $e^x \to +\infty$ as $x \to +\infty$, and $e^x \to 0$ as $x \to -\infty$. From this and the intermediate value theorem we see that the range of the function exp is $(0, \infty)$. Since exp is increasing, it is injective, and hence exp is a bijection from $\mathbf{R}$ to $(0, \infty)$, and thus has an inverse from $(0, \infty) \to \mathbf{R}$.

- **Definition** We define the *natural logarithm function* $\log : (0, \infty) \to \mathbf{R}$ (also called ln) to be the inverse of the exponential function. Thus $\exp(\log(x)) = x$ and $\log(\exp(x)) = x$.

- Since exp is continuous and strictly monotone increasing, we see that log is also continuous and strictly monotone increasing (see Proposition 3 of Week 7/8 notes of my Math 131AH class). Since exp is also differentiable, and the derivative is never zero, we see from the inverse function theorem that log is also differentiable (see Week 7/8 notes from Math 131AH). We list some other properties of the natural logarithm below.

- **Theorem 19**

- (a) For every $x \in (0, \infty)$, we have $\ln'(x) = \frac{1}{x}$. In particular, by the fundamental theorem of calculus, we have $\int_{[a,b]} \frac{1}{x}\, dx = \ln(b) - \ln(a)$ for any interval $[a, b]$ in $(0, \infty)$.

- (b) We have $\ln(xy) = \ln(x) + \ln(y)$ for all $x, y \in (0, \infty)$.

- (c) We have $\ln(1) = 0$ and $\ln(1/x) = -\ln(x)$ for all $x \in (0, \infty)$.

- (d) For any $x \in (0, \infty)$ and $y \in \mathbf{R}$, we have $\ln(x^y) = y \ln(x)$.

- (e) For any $x \in (-1, 1)$, we have

$$\ln(1 - x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}.$$

In particular, ln is analytic on $(0, 2)$, with the power series expansion

$$\ln(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x - 1)^n$$

for $x \in (0, 2)$.

- **Proof.** See Week 5 homework. $\square$

- We now give a modest application of Abel's theorem: from the alternating series test we see that $\sum_{n=1}^\infty \frac{(-1)^{n+1}}{n}$ is convergent. By Abel's theorem we thus see that

$$\sum_{n=1}^\infty \frac{(-1)^{n+1}}{n} = \lim_{x\to 2} \sum_{n=1}^\infty \frac{(-1)^{n+1}}{n}(x-1)^n$$
$$= \lim_{x\to 2} \ln(x) = \ln(2),$$

  thus we have the formula

$$\ln(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots.$$

* * * * *

A quick digression on complex numbers.

- To proceed further we need the complex number system $\mathbf{C}$. We assume that you are familiar with the complex numbers from other courses, so we will only give the briefest of descriptions here. Math 132 will give a much more detailed study of the analysis of complex numbers.

- **Definition.** A *complex number* is any expression of the form $a+bi$, where a and b are real numbers; the symbol i at present is just a placeholder with no intrinsic meaning. Two complex numbers $a+bi$ and $c+di$ are said to be equal, $a+bi = c+di$, iff $a=c$ and $b=d$. Every real number x is considered a complex number as well: $x = x+0i$. The sum of two complex numbers $a+bi$ and $c+di$ is defined as $(a+bi)+(c+di) := (a+c)+(b+d)i$. The product of two complex numbers is defined as $(a+bi)(c+di) := (ac-bd)+(ad+bc)i$. The difference of two complex numbers is defined by $(a+bi)-(c+di) := (a-c)+(b-d)i$. The quotient of two complex numbers is defined by $(a+bi)/(c+di) := (a+bi)(\frac{c}{c^2+d^2} - \frac{d}{c^2+d^2}i)$, provided that $c+di$ is non-zero. The *complex conjugate* of a complex number $a+bi$ is defined by $\overline{a+bi} := a-bi$. The absolute value of a complex number $a+bi$ is defined by $|a+bi| = \sqrt{a^2+b^2}$. The space of all complex numbers is called $\mathbf{C}$.

- We write i as shorthand for $0+1i$. Note in particular that $i^2 = -1$.

- We remark that the complex numbers obey all the normal rules of algebra, for instance if z, w, ζ are complex numbers, then $(z+w)\zeta = z\zeta + w\zeta$, and $(zw)\zeta = z(w\zeta)$, etc. (More precisely, the complex numbers form a *field*). One can also show that the rules of complex arithmetic are consistent with those of real arithmetic (e.g. $3+5$ is equal to 8 regardless of whether one uses the addition supplied by $\mathbf{R}$ or the addition supplied by $\mathbf{C}$).

- The operation of complex conjugation preserves all the arithmetic operations: $\overline{z+w} = \overline{z} + \overline{w}$, $\overline{z-w} = \overline{z} - \overline{w}$, $\overline{zw} = \overline{z}\,\overline{w}$, and $\overline{z/w} = \overline{z}/\overline{w}$. (In the language of algebra, conjugation is an *automorphism* of the complex numbers). The complex conjugate and absolute value are related by the identity $|z|^2 = z\overline{z}$.

- If z is a complex number, then $|z| = 0$ if and only if $z = 0$. One can show that $|zw| = |z||w|$, $|z/w| = |z|/|w|$ (if $w \neq 0$), and that $|z+w| \leq |z| + |w|$. In particular, we can turn $\mathbf{C}$ into a metric space by defining $d(z, w) := |z - w|$. One can show that $\mathbf{C}$ is in fact a complete metric space.

- The machinery of this set of notes and the previous set of notes of pointwise and uniform convergence of real-valued series of functions $\sum_{n=0}^\infty f(x)$ to cover complex-valued series as well. In fact there is almost no change in the theory. (In the textbook, complex-valued functions are used throughout).

- For instance, one can define the exponential function $\exp(z)$ for *complex* z in exactly the same manner as for real numbers:

$$\exp(z) := \sum_{n=0}^\infty \frac{z^n}{n!}.$$

One can state and prove the ratio test for complex series and use it to show that $\exp(z)$ converges for every z. It turns out that many of the properties from Theorem 17 still hold: we have that $\exp(z+w) = \exp(z) + \exp(w)$, for instance. (The other properties require complex differentiation and complex integration, and we will not discuss these here; those are topics for Math 132). Another useful observation is that $\overline{\exp(z)} = \exp(\overline{z})$; this can be seen by conjugating the partial sums $\sum_{n=0}^{N} \frac{z^n}{n!}$ and taking limits as $N \to \infty$.

- The complex logarithm turns out to be somewhat more subtle, mainly because exp is no longer invertible, and also because the various power series for the logarithm only have a finite radius of convergence (unlike exp, which has an infinite radius of convergence). Again, we will not discuss this rather delicate issue here and refer the reader to Math 132.

* * * * *

Trigonometric functions

- We now discuss the next most important class of special functions, after the exponential and logarithmic functions, namely the trigonometric functions. (There are several other useful special functions in mathematics, such as the hyperbolic trigonometric functions and hypergeometric functions, the gamma and zeta functions, and elliptic functions, but they occur more rarely and will not be discussed here.)

- Trigonometric functions are often defined using geometric concepts, notably those of circles, triangles, and angles. However, it is also possible to define them using more analytic concepts, and in particular the (complex) exponential function.

- **Definition** If x is a real number, then we define

$$\cos(x) := \frac{e^{ix} + e^{-ix}}{2}$$

and

$$\sin(x) := \frac{e^{ix} - e^{-ix}}{2i}.$$

We refer to cos and sin as the *cosine* and *sine* functions respectively.

- These formulae were discovered by Euler, who recognized the link between the complex exponential and the trigonometric functions.

- From the power series definition of exp, we have

$$e^{ix} = 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \dots$$

and

$$e^{-ix} = 1 - ix - \frac{x^2}{2!} + \frac{ix^3}{3!} + \frac{x^4}{4!} - \dots$$

and so from the above formulae we have

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

and

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}.$$

- In particular, $\cos(x)$ and $\sin(x)$ are always real. Clearly the functions sin and cos are also real-analytic on $(-\infty, \infty)$, since their power series converges for every x (e.g. by the ratio test). In particular the sine and cosine functions are continuous and differentiable.

- We list some basic properties of the sine and cosine functions below.

- **Theorem 20.**

- (a) For any real number x, we have $\sin(x)^2 + \cos(x)^2 = 1$. In particular, we have $\sin(x) \in [-1, 1]$ and $\cos(x) \in [-1, 1]$ for all $x \in \mathbf{R}$.

- (b) For any real number x, we have $\sin'(x) = \cos(x)$ and $\cos'(x) = -\sin(x)$.

- (c) For any real number x, we have $\sin(-x) = -\sin(x)$ and $\cos(-x) = \cos(x)$.

- (d) For any real numbers x, y, we have $\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$ and $\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$.

- (e) We have $\sin(0) = 0$ and $\cos(0) = 1$.

- (f) For every real number x, we have $e^{ix} = \cos(x) + i\sin(x)$ and $e^{-ix} = \cos(x) - i\sin(x)$.

- **Proof.** See Week 5 homework. $\square$

- Now we describe some other properties of sin and cos.

- **Lemma 21.** There exists a positive number x such that $\sin(x) = 0$.

- **Proof.** Suppose for contradiction that $\sin(x) \neq 0$ for all $x \in (0, \infty)$. Observe that this would also imply that $\cos(x) \neq 0$ for all $x \in (0, \infty)$, since if $\cos(x) = 0$ then $\sin(2x) = 0$ by Theorem 20(d) (why?). Since $\cos(0) = 1$, this implies by the intermediate value theorem that $\cos(x) > 0$ for all $x > 0$ (why?). Also, since $\sin(0) = 0$ and $\sin'(0) = 1 > 0$, we see that sin increasing near 0, hence is positive to the right of 0. By the intermediate value theorem we again conclude that $\sin(x) > 0$ for all $x > 0$ (otherwise sin would have a zero on $(0, \infty)$).

- In particular if we define the cotangent function $\cot(x) := \cos(x)/\sin(x)$, then $\cot(x)$ would be positive and differentiable on all of $(0, \infty)$. From the quotient rule and Theorem 20 we see that the derivative of $\cot(x)$ is $-1/\sin(x)^2$ (why?) In particular, we have $\cot'(x) \leq -1$ for all $x > 0$. By the fundamental theorem of calculus this implies that $\cot(x + s) \leq \cot(x) - s$ for all $x > 0$ and $s > 0$. But letting $s \to \infty$ we see that this contradicts our assertion that cot is positive on $(0, \infty)$ (why?). $\square$

- Let E be the set $E := \{x \in (0, \infty) : \sin(x) = 0\}$, i.e. E is the set of roots of sin on $(0, \infty)$. By Lemma 21, E is non-empty. Also, since sin is continuous, E is closed (why? use Theorem 13(d) from Week 2 notes). Thus E contains all its adherent points, and thus contains $\inf(E)$. Thus if we make the definition

- **Definition** We define π to be the number

$$\pi := \inf\{x \in (0, \infty) : \sin(x) = 0\}$$

then we have $\pi > 0$ and $\sin(\pi) = 0$. By definition of π, sin cannot have any zeroes in $(0, \pi)$, and so in particular must be positive on

$(0, \pi)$, (cf. the arguments in Lemma 21 using the intermediate value theorem). Since $\cos'(x) = -\sin(x)$, we thus conclude that $\cos(x)$ is strictly decreasing on $(0, \pi)$. Since $\cos(0) = 1$, this implies in particular that $\cos(\pi) < 1$; since $\sin^2(\pi) + \cos^2(\pi) = 1$ and $\sin(\pi) = 0$, we thus conclude that $\cos(\pi) = -1$.

- In particular we have Euler's famous formula

$$e^{\pi i} = \cos(\pi) + i \sin(\pi) = -1.$$

- We now conclude with some other properties of sine and cosine.

- **Theorem 22.**

- (a) For any real x we have $\cos(x + \pi) = -\cos(x)$ and $\sin(x + \pi) = -\sin(x)$. In particular we have $\cos(x + 2\pi) = \cos(x)$ and $\sin(x + 2\pi) = \sin(x)$, i.e. sin and cos are periodic with period 2π.

- (b) If x is real, then $\sin(x) = 0$ if and only if x/π is an integer.

- (c) If x is real, then $\cos(x) = 0$ if and only if x/π is an integer plus $1/2$.

- **Proof.** See Week 5 homework. $\square$

- We can of course define all the other trigonometric functions: tangent, cotangent, secant, and cosecant, and develop all the familiar identities of trigonometry. But we will not do so here (since you presumably know how to do all this anyway).