Math 131BH - Week 10
Textbook pages covered: 314-324

- The Lebesgue integral
- Properties of the Lebesgue integral
- Comparison with the Riemann integral
- Dominated convergence theorem

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Integration of simple functions

- In last week's notes, we defined the notion of a simple function - a measurable function which takes finitely many values. We now show how to integrate these functions, at least when the simple function is non-negative. Then we will integrate measurable non-negative functions, and finally integrate general measurable functions (or at least the absolutely integrable ones).

- **Definition** Let Ω be a measurable subset of $\mathbf{R}^n$, and let $f : \Omega \to \mathbf{R}$ be a simple function which is non-negative; thus f is measurable and the image $f(\Omega)$ is finite and contained in $[0, \infty)$. We then define the *Lesbegue integral* $\int_\Omega f$ of f on Ω by

$$\int_\Omega f := \sum_{\lambda \in f(\Omega); \lambda > 0} \lambda m(\{x \in \Omega : f(x) = \lambda\}).$$

- We also write $\int_\Omega f$ as $\int_\Omega f\ dm$ (to emphasize the role of Lebesgue measure m) or use a dummy variable such as x, e.g. $\int_\Omega f(x)\ dx$.

- **Example.** Let $f : \mathbf{R} \to \mathbf{R}$ be the function which equals 3 on the interval $[1, 2]$, equals 4 on the integral $(2, 4)$, and is zero everywhere else. Then

$$\int_\Omega f := 3 \times m([1, 2]) + 4 \times m((2, 4)) = 3 \times 1 + 4 \times 2 = 11.$$

Or if $g : \mathbf{R} \to \mathbf{R}$ is the function which equals 1 on $[0, \infty)$ and is zero everywhere else, then

$$\int_\Omega g = 1 \times m([0, \infty)) = 1 \times +\infty = +\infty.$$

Thus the simple integral of a simple function can equal $+\infty$. (The reason why we restrict this integral to non-negative functions is to avoid ever encountering the indefinite form $+\infty + (-\infty)$).

- Note that this definition of integral corresponds to one's intuitive notion of integration (at least of non-negative functions) as the area under the graph of the function (or volume, if one is in higher dimensions).

- Another formulation of the integral for non-negative simple functions is as follows.

- **Lemma 1.** Let Ω be a measurable subset of $\mathbf{R}^n$, and let $E_1, \ldots, E_N$ are a finite number of disjoint measurable subsets in Ω. Let $c_1, \ldots, c_N$ be non-negative numbers (not necessarily distinct). Then we have

$$\int_\Omega \sum_{j=1}^N c_j \chi_{E_j} = \sum_{j=1}^n c_j m(E_j).$$

- **Proof.** We can assume that none of the c_j are zero, since we can just remove them from the sum on both sides of the equation. Let $f := \sum_{j=1}^N c_j \chi_{E_j}$. Then $f(x)$ is either equal to one of the c_j (if $x \in E_j$) or equal to 0 (if $x \notin \bigcup_{j=1}^N E_j$). Thus f is a simple function, and $f(\Omega) = \{0\} \cup \{c_j : 1 \leq j \leq N\}$. Thus, by the definition,

$$\int_\Omega f = \sum_{\lambda \in \{c_j : 1 \leq j \leq N\}} \lambda m(\{x \in \Omega : f(x) = \lambda\})$$

$$= \sum_{\lambda \in \{c_j : 1 \leq j \leq N\}} \lambda m(\bigcup_{1 \leq j \leq N : c_j = \lambda} E_j).$$

But by the finite additivity property of Lebesgue measure, this is equal to

$$\sum_{\lambda \in \{c_j : 1 \leq j \leq N\}} \lambda \sum_{1 \leq j \leq N : c_j = \lambda} m(E_j)$$

$$\sum_{\lambda \in \{c_j : 1 \leq j \leq N\}} \sum_{1 \leq j \leq N : c_j = \lambda} c_j m(E_j).$$

Each j appears exactly once in this sum, since c_j is only equal to exactly one value of λ. So the above expression is equal to $\sum_{1 \leq j \leq N} c_j m(E_j)$ as desired. $\square$

- Some basic properties of Lebesgue integration of non-negative simple functions:

- **Proposition 2.** Let Ω be a measurable set, and let $f : \Omega \to \mathbf{R}$ and $g : \Omega \to \mathbf{R}$ be non-negative simple functions.

- (a) We have $0 \leq \int_\Omega f \leq \infty$. Furthermore, we have $\int_\Omega f = 0$ if and only if $m(\{x \in \Omega : f(x) \neq 0\}) = 0$.

- (b) We have $\int_\Omega (f+g) = \int_\Omega f + \int_\Omega g$.

- (c) For any positive number c, we have $\int_\Omega cf = c \int_\Omega f$.

- (d) If $f(x) \leq g(x)$ for all $x \in \Omega$, then we have $\int_\Omega f \leq \int_\Omega g$.

- We make a very convenient notational convention: if a property $P(x)$ holds for all points in Ω, except for a set of measure zero, then we say that P holds for *almost every* point in Ω. Thus (a) asserts that $\int_\Omega f = 0$ if and only if f is zero for almost every point in Ω.

- **Proof.** From Lemma 23 from last week's notes, or from the formula

$$f = \sum_{\lambda \in f(\Omega) \backslash \{0\}} \lambda \chi_{\{x \in \Omega : f(x) = \lambda\}}$$

we can write f as a combination of characteristic functions, say

$$f = \sum_{j=1}^{N} c_j \chi_{E_j},$$

where $E_1, \ldots, E_N$ are disjoint subsets of Ω and the c_j are positive. Similarly we can write

$$g = \sum_{k=1}^{M} d_k \chi_{F_k}$$

where $F_1, \ldots, F_M$ are disjoint subsets of Ω and the d_k are positive.

- (a) Since $\int_\Omega f = \sum_{j=1}^N c_j m(E_j)$ it is clear that the integral is between 0 and infinity. If f is zero almost everywhere, then all of the E_j must have measure zero (why?) and so $\int_\Omega f = 0$. Conversely, if $\int_\Omega f = 0$, then $\sum_{j=1}^N c_j m(E_j) = 0$, which can only happen when all of the $m(E_j)$ are zero (since all the c_j are positive). But then $\bigcup_{j=1}^N E_j$ has measure zero, and hence f is zero almost everywhere in Ω.

- (b) Write $E_0 := \Omega \backslash \bigcup_{j=1}^N E_j$ and $c_0 := 0$, then we have $\Omega = E_0 \cup E_1 \cup \ldots \cup E_N$ and

$$f = \sum_{j=0}^N c_j \chi_{E_j}.$$

Similarly if we write $F_0 := \Omega \backslash \bigcup_{k=1}^M F_k$ and $d_0 := 0$ then

$$g = \sum_{k=0}^M d_k \chi_{F_k}.$$

Since $\Omega = E_0 \cup \ldots \cup E_N = F_0 \cup \ldots \cup F_M$, we have

$$f = \sum_{j=0}^N \sum_{k=0}^M c_j \chi_{E_j \cap F_k}$$

and

$$g = \sum_{k=0}^M \sum_{j=0}^N d_k \chi_{E_j \cap F_k}$$

and hence

$$f + g = \sum_{0 \leq j \leq N; 0 \leq k \leq M} (c_j + d_k) \chi_{E_j \cap F_k}.$$

By Lemma 1, we thus have

$$\int_\Omega (f+g) = \sum_{0 \leq j \leq N; 0 \leq k \leq M} (c_j + d_k) m(E_j \cap F_k).$$

On the other hand, we have

$$\int_\Omega f = \sum_{0 \leq j \leq N} c_j m(E_j) = \sum_{0 \leq j \leq N; 0 \leq k \leq M} c_j m(E_j \cap F_k)$$

and similarly

$$\int_\Omega g = \sum_{0 \leq k \leq M} d_k m(F_k) = \sum_{0 \leq j \leq N; 0 \leq k \leq M} d_k m(E_j \cap F_k)$$

and the claim (b) follows.

- (c) Since $cf = \sum_{j=1}^N cc_j \chi_{E_j}$, we have $\int_\Omega cf = \sum_{j=1}^N cc_j m(E_j)$. Since $\int_\Omega f = \sum_{j=1}^N c_j m(E_j)$, the claim follows.

- (d) Write $h := g - f$. Then h is simple and non-negative and $g = f + h$, hence by (b) we have $\int_\Omega g = \int_\Omega f + \int_\Omega h$. But by (a) we have $\int_\Omega h \geq 0$, and the claim follows. $\square$

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Integration of non-negative measurable functions

- We now pass from the integration of non-negative simple functions to the integration of non-negative measurable functions. We will allow our measurable functions to take the value of $+\infty$ sometimes.

- **Definition.** Let $f : \Omega \to \mathbf{R}$ and $g : \Omega \to \mathbf{R}$ be functions. We say that f *majorizes* g, or g *minorizes* f, iff we have $f(x) \geq g(x)$ for all $x \in \Omega$.

- We sometimes use the phrase "f dominates g" instead of "f majorizes g".

- **Definition.** Let Ω be a measurable subset of $\mathbf{R}^n$, and let $f : \Omega \to [0, \infty]$ be measurable and non-negative. Then we define the *Lebesgue integral* $\int_\Omega f$ of f on Ω to be

$$\int_\Omega f := \sup\{\int_\Omega s : s \text{ is simple and non-negative, and minorizes } f\}.$$

- The reader should compare this notion to that of a lower Riemann integral from my Math 131AH notes. Interestingly, we will not need to match this lower integral with an upper integral here.

- Note that if Ω' is any measurable subset of Ω, then we can define $\int_{\Omega'} f$ as well by restricting f to Ω', thus $\int_{\Omega'} f := \int_{\Omega'} f|_{\Omega'}$.

- We have to check that this definition is consistent with our previous notion of Lebesgue integral for non-negative simple functions; in other words, if $f : \Omega \to \mathbf{R}$ is a non-negative simple function, then the value of $\int_\Omega f$ given by this definition should be the same as the one given in the previous definition. But this is clear because f certainly minorizes itself, and any other non-negative simple function s which minorizes f will have an integral $\int_\Omega s$ less than or equal to $\int_\Omega f$, thanks to Proposition 2(d).

- Also, note that $\int_\Omega f$ is always at least 0, since 0 is simple, non-negative, and minorizes f. Of course, $\int_\Omega f$ could equal $+\infty$.

- Some basic properties of the Lebesgue integral on non-negative measurable functions (which supercede Proposition 2):

- **Proposition 3.** Let Ω be a measurable set, and let $f : \Omega \to [0, \infty]$ and $g : \Omega \to [0, \infty]$ be non-negative measurable functions.

- (a) We have $0 \leq \int_\Omega f \leq \infty$. Furthermore, we have $\int_\Omega f = 0$ if and only if $f(x) = 0$ for almost every $x \in \Omega$.

- (b) For any positive number c, we have $\int_\Omega cf = c \int_\Omega f$.

- (c) If $f(x) \leq g(x)$ for all $x \in \Omega$, then we have $\int_\Omega f \leq \int_\Omega g$.

- (d) If $f(x) = g(x)$ for almost every $x \in \Omega$, then $\int_\Omega f = \int_\Omega g$.

- (e) If $\Omega' \subseteq \Omega$ is measurable, then $\int_{\Omega'} f = \int_\Omega f \chi_{\Omega'} \leq \int_\Omega f$.

- **Proof.** See Week 9 homework. □

- Proposition 3(d) is quite interesting; it says that one can modify the values of a function on any measure zero set (e.g. you can modify a function on every rational number), and not affect its integral at all. It is as if no individual point, or even a measure zero collection of points, has any "vote" in what the integral of a function should be; only the collective set of points has an influence on an integral.

- Note that we do not yet try to interchange sums and integrals. From the definition it is fairly easy to prove that $\int_\Omega (f + g) \geq \int_\Omega f + \int_\Omega g$, but to prove equality requires more work and will be done later.

- As we have seen in previous weeks notes, we cannot always interchange an integral with a limit (or with limit-like concepts such as supremum). However, with the Lebesgue integral it is possible to do so if the functions are increasing:

- **Lebesgue monotone convergence theorem.** Let Ω be a measurable subset of $\mathbf{R}^n$, and let $(f_n)_{n=1}^\infty$ be a sequence of non-negative measurable functions from Ω to $\mathbf{R}$ which are increasing in the sense that

$$0 \leq f_1(x) \leq f_2(x) \leq f_3(x) \leq \ldots \text{ for all } x \in \Omega.$$

(Note we are assuming that $f_n(x)$ is increasing with respect to n; this is a different notion from $f_n(x)$ increasing with respect to x). Then we have

$$0 \leq \int_\Omega f_1 \leq \int_\Omega f_2 \leq \int_\Omega f_3 \leq \ldots$$

and

$$\int_\Omega \sup_n f_n = \sup_n \int_\Omega f_n.$$

- **Proof.** The first conclusion is clear from Proposition 3(c). Now we prove the second conclusion. From Proposition 3(c) again we have

$$\int_\Omega \sup_m f_m \geq \int_\Omega f_n$$

for every n; taking suprema in n we obtain

$$\int_\Omega \sup_m f_m \geq \sup_n \int_\Omega f_n$$

which is one half of the desired conclusion. To finish the proof we have to show

$$\int_\Omega \sup_m f_m \leq \sup_n \int_\Omega f_n.$$

From the definition of $\int_\Omega \sup_m f_m$, it will suffice to show that

$$\int_\Omega s \leq \sup_n \int_\Omega f_n$$

for all simple non-negative functions which minorize $\sup_m f_m$.

- Fix s. We will show that

$$(1-\varepsilon)\int_\Omega s \leq \sup_n \int_\Omega f_n$$

for every $0 < \varepsilon < 1$; the claim then follows by taking limits as $\varepsilon \to 0$.

- Fix ε. By construction of s, we have

$$s(x) \leq \sup_n f_n(x)$$

for every $x \in \Omega$. Hence, for every $x \in \Omega$ there exists an N (depending on x) such that

$$f_N(x) \geq (1-\varepsilon)s(x).$$

Since the f_n are increasing, this will imply that $f_n(x) \geq (1-\varepsilon)s(x)$ for all $n \geq N$. Thus, if we define the sets E_n by

$$E_n := \{x \in \Omega : f_n(x) \geq (1-\varepsilon)s(x)\}$$

then we have $E_1 \subset E_2 \subset E_3 \subset \ldots$ and $\bigcup_{n=1}^\infty E_n = \Omega$.

- From Proposition 3(cdf) we have

$$(1-\varepsilon)\int_{E_n} s = \int_{E_n}(1-\varepsilon)s \leq \int_{E_n} f_n \leq \int_\Omega f_n$$

so to finish the argument it will suffice to show that

$$\sup_n \int_{E_n} s = \int_\Omega s.$$

Since s is a simple function, we may write $s = \sum_{j=1}^N c_j \chi_{F_j}$ for some measurable F_j and positive c_j. Since

$$\int_\Omega s = \sum_{j=1}^N c_j m(F_j)$$

and

$$\int_{E_n} s = \int_{E_n} \sum_{j=1}^N c_j \chi_{F_j \cap E_n} = \sum_{j=1}^N c_j m(F_j \cap E_n)$$

it thus suffices to show that

$$\sup_n m(F_j \cap E_n) = m(F_j)$$

for each j. But this follows from Homework 1(a) of Assignment 8. $\square$

- This theorem is extremely useful. For instance, we can now do addition:

- **Lemma 4.** Let Ω be a measurable subset of $\mathbf{R}^n$, and let $f : \Omega \to [0, \infty]$ and $g : \Omega \to [0, \infty]$ be measurable functions. Then $\int_\Omega (f + g) = \int_\Omega f + \int_\Omega g$.

- **Proof.** By Lemma 24 of last week's notes, there exists a sequence $0 \leq s_1 \leq s_2 \leq \ldots \leq f$ of simple functions such that $\sup_n s_n = f$, and similarly a sequence $0 \leq t_1 \leq t_2 \leq \ldots \leq g$ of simple functions such that $\sup_n t_n = g$. Since the s_n are increasing and the t_n are increasing, it is then easy to check that $s_n + t_n$ is also increasing and $\sup_n (s_n + t_n) = f + g$ (why?). By the monotone convergence theorem we thus have

$$\int_\Omega f = \sup_n \int_\Omega s_n$$

$$\int_\Omega g = \sup_n \int_\Omega t_n$$

$$\int_\Omega (f + g) = \sup_n \int_\Omega (s_n + t_n).$$

But by Proposition 1(b) we have $\int_\Omega (s_n + t_n) = \int_\Omega s_n + \int_\Omega t_n$. By Proposition 1(d), $\int_\Omega s_n$ and $\int_\Omega t_n$ are both increasing in n, so

$$\sup_n (\int_\Omega s_n + \int_\Omega t_n) = (\sup_n \int_\Omega s_n) + (\sup_n \int_\Omega t_n)$$

and the claim follows. $\square$

- Of course, once one can interchange an integral with a sum of two functions, one can handle an integral and any finite number of functions by induction. More surprisingly, one can handle infinite sums as well of *non-negative* functions:

- **Corollary 5.** If Ω is a measurable subset of $\mathbf{R}^n$, and $g_1, g_2, \ldots$ are a sequence of non-negative functions from Ω to $[0, \infty]$, then

$$\int_\Omega \sum_{n=1}^\infty g_n = \sum_{n=1}^\infty \int_\Omega g_n.$$

- **Proof.** Apply the monotone convergence theorem with $f_N := \sum_{n=1}^N g_n$. $\square$

- Note that we do not need to assume anything about the convergence of the above sums; it may well happen that both sides are equal to $+\infty$. However, we *do* need to assume non-negativity; see homework.

- One could similarly ask whether we could interchange limits and integrals; in other words, is it true that

$$\int_\Omega \lim_{n\to\infty} f_n = \lim_{n\to\infty} \int_\Omega f_n.$$

Unfortunately, this is not true, as the following "moving bump" example shows. For each $n = 1, 2, 3 \ldots$, let $f_n : \mathbf{R} \to \mathbf{R}$ be the function $f_n = \chi_{[n,n+1)}$. Then $\lim_{n\to\infty} f_n(x) = 0$ for every x, but $\int_{\mathbf{R}} f_n = 1$ for every n, and hence $\lim_{n\to\infty} \int_{\mathbf{R}} f_n = 1 \neq 0$. In other words, the limiting function $\lim_{n\to\infty} f_n$ can end up having significantly smaller integral than any of the original integrals. However, the following very useful lemma of Fatou shows that the reverse cannot happen - there is no way the limiting function has larger integral than the (limit of the) original integrals:

- **Fatou's lemma** Let Ω be a measurable subset of $\mathbf{R}^n$, and let $f_1, f_2, \ldots$ be a sequence of non-negative functions from Ω to $[0, \infty]$. Then

$$\int_\Omega \liminf_{n\to\infty} f_n \leq \liminf_{n\to\infty} \int_\Omega f_n.$$

- **Proof.** Recall that

$$\liminf_{n\to\infty} f_n = \sup_n (\inf_{m\geq n} f_m)$$

and hence by the monotone convergence theorem

$$\int_\Omega \liminf_{n\to\infty} f_n = \sup_n \int_\Omega (\inf_{m\geq n} f_m).$$

By Proposition 3(c) we have

$$\int_\Omega (\inf_{m\geq n} f_m) \leq \int_\Omega f_j$$

for every $j \geq n$; taking infima in j we obtain

$$\int_\Omega (\inf_{m\geq n} f_m) \leq \inf_{j\geq n} \int_\Omega f_j.$$

Thus

$$\int_\Omega \liminf_{n\to\infty} f_n \leq \sup_n \inf_{j\geq n} \int_\Omega f_j = \liminf_{n\to\infty} \int_\Omega f_n$$

as desired. □

- Note that we are allowing our functions to take the value $+\infty$ at some points. It is even possible for a function to the value $+\infty$ but still have a finite integral; for instance, if E is a measure zero set, and $f : \Omega \to \mathbf{R}$ is equal to $+\infty$ on E but equals 0 everywhere else, then $\int_\Omega f = 0$ by Proposition 3(a). However, if the integral is finite, the function must be finite almost everywhere:

- **Lemma 6.** Let Ω be a measurable subset of $\mathbf{R}^n$, and let $f : \Omega \to [0, \infty]$ be a non-negative measurable function such that $\int_\Omega f$ is finite. Then f is finite almost everywhere (i.e. the set $\{x \in \Omega : f(x) = +\infty\}$ has measure zero).

- **Proof.** See Week 9 homework. □

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Integration of absolutely integrable functions

- We have now completed the theory of the Lebesgue integral for non-negative functions. Now we consider how to integrate functions which can be both positive and negative. However, we do wish to avoid the indefinite expression $+\infty + (-\infty)$, so we will restrict our attention to a subclass of measurable functions - the *absolutely integrable functions*.

- **Definition.** Let Ω be a measurable subset of $\mathbf{R}^n$. A measurable function $f : \Omega \to \mathbf{R}^*$ is said to be *absolutely integrable* if the integral $\int_\Omega |f|$ is finite.

- Of course, $|f|$ is always non-negative, so this definition makes sense even if f changes sign. Absolutely integrable functions are also known as $L^1(\Omega)$ functions.

- If $f : \Omega \to \mathbf{R}^*$ is a function, we define the *positive part* $f^+ : \Omega \to [0, \infty]$ and *negative part* $f^- : \Omega \to [0, \infty]$ by the formulae

$$f^+ := \max(f, 0); \quad f^- := -\min(f, 0).$$

- From Corollary 18 from last week's notes we know that f^+ and f^- are measurable. Observe also that f^+ and f^- are non-negative, that $f = f^+ - f^-$, and $|f| = f^+ + f^-$. (Why?).

- **Definition.** Let $f : \Omega \to \mathbf{R}^*$ be an absolutely integrable function. We define the *Lebesgue integral* $\int_\Omega f$ of f to be the quantity

$$\int_\Omega f := \int_\Omega f^+ - \int_\Omega f^-.$$

- Note that since f is absolutely integrable, $\int_\Omega f^+$ and $\int_\Omega f^-$ are less than or equal to $\int_\Omega |f|$ and hence are finite. Thus $\int_\Omega f$ is always finite; we are never encountering the indeterminate form $+\infty - (+\infty)$.

- Note that this definition is consistent with our previous definition of the Lebesgue integral for non-negative functions, since if f is non-negative then $f^+ = f$ and $f^- = 0$. Clearly, we also have the useful *triangle inequality*

$$|\int_\Omega f| \leq \int_\Omega f^+ + \int_\Omega f^- = \int_\Omega |f|$$

(why is this true?).

- Some other properties of the Lebesgue integral:

- **Proposition 7.** Let Ω be a measurable set, and let $f : \Omega \to \mathbf{R}$ and $g : \Omega \to \mathbf{R}$ be absolutely integrable functions.

- (a) For any real number c (positive, zero, or negative), we have that cf is absolutely integrable and $\int_\Omega cf = c\int_\Omega f$.

- (b) The function $f+g$ is absolutely integrable, and $\int_\Omega (f+g) = \int_\Omega f + \int_\Omega g$.

- (c) If $f(x) \leq g(x)$ for all $x \in \Omega$, then we have $\int_\Omega f \leq \int_\Omega g$.

- (d) If $f(x) = g(x)$ for almost every $x \in \Omega$, then $\int_\Omega f = \int_\Omega g$.

- **Proof.** See Week 9 homework. $\square$

- As mentioned in the previous section, one cannot necessarily interchange limits and integrals, $\lim \int f_n = \int \lim f_n$, as the "moving bump example" showed. However, it is possible to exclude the moving bump example, and successfully interchange limits and integrals: if we know that the functions f_n are all majorized by a single absolutely integrable:

- **Lebesgue dominated convergence theorem.** Let Ω be a measurable subset of $\mathbf{R}^n$, and let $f_1, f_2, \ldots$ be a sequence of measurable functions from Ω to $\mathbf{R}^*$ which converge pointwise. Suppose also that there is an absolutely integrable function $F : \Omega \to [0, \infty]$ such that $|f_n(x)| \leq F(x)$ for all $x \in \Omega$ and all $n = 1, 2, 3, \ldots$. Then

$$\int_\Omega \lim_{n\to\infty} f_n = \lim_{n\to\infty} \int_\Omega f_n.$$

- **Proof.** Let $f : \Omega \to \mathbf{R}^*$ be the function $f(x) := \lim_{n\to\infty} f_n(x)$; this function exists by hypothesis. By Lemma 21 from last week's notes, f is measurable. Also, since $|f_n(x)| \leq F(x)$ for all n and all $x \in \Omega$, we see that each f_n is absolutely integrable, and by taking limits we obtain $|f(x)| \leq F(x)$ for all $x \in \Omega$, so f is also absolutely integrable. Our task is to show that $\lim_{n\to\infty} \int_\Omega f_n = \int_\Omega f$.

- The functions $F + f_n$ are non-negative and converge pointwise to $F + f$. So by Fatou's lemma

$$\int_\Omega F + f \leq \liminf_{n\to\infty} \int_\Omega F + f_n$$

and thus

$$\int_\Omega f \le \liminf_{n\to\infty} \int_\Omega f_n.$$

But the functions $F - f_n$ are also non-negative and converge pointwise to $F - f$. So by Fatou's lemma again

$$\int_\Omega F - f \le \liminf_{n\to\infty} \int_\Omega F - f_n.$$

Since the right-hand side is $\int_\Omega F - \limsup_{n\to\infty} \int_\Omega f_n$ (why did the lim inf become a lim sup?), we thus have

$$\int_\Omega f \ge \limsup_{n\to\infty} \int_\Omega f_n.$$

Thus the lim inf and lim sup of $\int_\Omega f_n$ are both equal to $\int_\Omega f$, as desired. $\square$

- Finally, we record a lemma which is not particularly interesting in itself, but will have some useful consequences later in these notes.

- **Definition.** Let Ω be a measurable subset of $\mathbf{R}^n$, and let $f : \Omega \to \mathbf{R}$ be a function (not necessarily measurable). We define the *upper Lebesgue integral* $\overline{\int}_\Omega f$ to be

$$\overline{\int}_\Omega f := \inf\{\int_\Omega g : g \text{ is an absolutely integrable function from } \Omega \text{ to } \mathbf{R} \text{ that majorizes } f\}$$

and the *lower Lebesgue integral* $\underline{\int}_\Omega f$ to be

$$\underline{\int}_\Omega f := \sup\{\int_\Omega g : g \text{ is an absolutely integrable function from } \Omega \text{ to } \mathbf{R} \text{ that minorizes } f\}.$$

- It is easy to see that $\underline{\int}_\Omega f \le \overline{\int}_\Omega f$ (why? use Proposition 7(c)). When f is absolutely integrable then equality occurs (why?). The converse is also true:

- **Lemma 8.** Let Ω be a measurable subset of $\mathbf{R}^n$, and let $f : \Omega \to \mathbf{R}$ be a function (not necessarily measurable). Let A be a real number, and suppose $\overline{\int}_\Omega f = \underline{\int}_\Omega f = A$. Then f is absolutely integrable, and

$$\int_\Omega f = \overline{\int}_\Omega f = \underline{\int}_\Omega f = A.$$

- **Proof.** By definition of upper Lebesgue integral, for every integer $n \geq 1$ we may find an absolutely integrable function $f_n^+ : \Omega \to \mathbf{R}$ which majorizes f such that

$$\int_\Omega f_n^+ \leq A + \frac{1}{n}.$$

Similarly we may find an absolutely integrable function $f_n^- : \Omega \to \mathbf{R}$ which minorizes f such that

$$\int_\Omega f_n^- \leq A - \frac{1}{n}.$$

Let $F^+ := \inf_n f_n^+$ and $F^- := \sup_n f_n^-$. Then F^+ and F^- are measurable (by Lemma 21) and absolutely integrable (because they are squeezed between the absolutely integrable functions f_1^+ and f_1^-, for instance). Also, F^+ majorizes f and F^- minorizes f. Finally, we have

$$\int_\Omega F^+ \leq \int_\Omega f_n^+ \leq A + \frac{1}{n}$$

for every n, and hence

$$\int_\Omega F^+ \leq A.$$

Similarly we have

$$\int_\Omega F^- \geq A.$$

but F^+ majorizes F^-, and hence $\int_\Omega F^+ \geq \int_\Omega F^-$. Hence we must have

$$\int_\Omega F^+ = \int_\Omega F^- = A.$$

In particular
$$\int_\Omega F^+ - F^- = 0.$$
By Proposition 3(a), we thus have $F^+(x) = F^-(x)$ for almost every x. But since f is squeezed between F^- and F^+, we thus have $f(x) = F^+(x) = F^-(x)$ for almost every x. In particular, f differs from the absolutely integrable function F^+ only on a set of measure zero and is thus easily shown to be measurable (note that any subset of a measure zero set is also measurable with measure zero - why?) and absolutely integrable, with
$$\int_\Omega f = \int_\Omega F^+ = \int_\Omega F^- = A$$
as desired. $\square$

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Comparison with the Riemann integral

- We have spent a lot of effort constructing the Lebesgue integral, but have not yet addressed the question of how to actually compute any Lebesgue integrals, and whether Lebesgue integration is any different from the Riemann integral (say for integrals in one dimension). Now we show that the Lebesgue integral is a generalization of the Riemann integral. To clarify matters, we shall temporarily distinguish the Riemann integral from the Lebesgue integral by writing the Riemann integral $\int_I f$ as $R. \int_I f$.

- Our objective here is to prove

- **Proposition 9.** Let $I \subseteq \mathbf{R}$ be an interval, and let $f : I \to \mathbf{R}$ be a Riemann integrable function. Then f is also absolutely integrable, and $\int_I f = R. \int_I f$.

- **Proof.** (Optional) Write $A := R. \int_I f$. Since f is Riemann integrable, we know that the upper and lower Riemann integrals are equal to A. Thus, for every $\varepsilon > 0$, there exists a partition $\mathbf{P}$ of I into smaller intervals J such that
$$A - \varepsilon \leq \sum_{J \in \mathbf{P}} |J| \inf_{x \in J} f(x) \leq A \leq \sum_{J \in \mathbf{P}} |J| \sup_{x \in J} f(x) \leq A + \varepsilon,$$

where $|J|$ denotes the length of J. Note that $|J|$ is the same as $m(J)$, since J is a box.

- Let $f_\varepsilon^- : I \to \mathbf{R}$ and $f_\varepsilon^+ : I \to \mathbf{R}$ be the functions

$$f_\varepsilon^-(x) = \sum_{J \in \mathbf{P}} \inf_{x \in J} f(x) \chi_J(x)$$

and

$$f_\varepsilon^+(x) = \sum_{J \in \mathbf{P}} \sup_{x \in J} f(x) \chi_J(x);$$

these are simple functions and hence measurable and absolutely integrable. By Lemma 1 we have

$$\int_I f_\varepsilon^- = \sum_{J \in \mathbf{P}} |J| \inf_{x \in J} f(x)$$

and

$$\int_I f_\varepsilon^+ = \sum_{J \in \mathbf{P}} |J| \sup_{x \in J} f(x)$$

and hence

$$A - \varepsilon \leq \int_I f_\varepsilon^- \leq A \leq \int_I f_\varepsilon^+ \leq A + \varepsilon.$$

Since f_ε^+ majorizes f, and f_ε^- minorizes f, we thus have

$$A - \varepsilon \leq \underline{\int}_\Omega f \leq \overline{\int}_\Omega f \leq A + \varepsilon$$

for every ε, and thus

$$\underline{\int}_\Omega f = \overline{\int}_\Omega f = A$$

and hence by Lemma 8, f is absolutely integrable with $\int_I f = A$, as desired. $\square$

- Thus every Riemann integrable function is also Lebesgue integrable, at least on bounded intervals, and we no longer need the $R. \int_I f$ notation. However, the converse is not true. Take for instance the function f :

$[0,1] \to \mathbf{R}$ defined by $f(x) := 1$ when x is rational, and $f(x) := 0$ when x is irrational. Then it is easy to see that the upper Riemann integral of f on $[0,1]$ is 1, and the lower Riemann integral is 0, so f is not Riemann integrable on f. On the other hand, f is the characteristic function of the set $\mathbf{Q} \cap [0,1]$, which is countable and hence measure zero. Thus f is Lebesgue integrable and $\int_{[0,1]} f = 0$. Thus the Lebesgue integral can handle more functions than the Riemann integral; this is one of the primary reasons why we use the Lebesgue integral in analysis. (The other reason is that the Lebesgue integral interacts well with limits, as the Lebesgue monotone convergence theorem, Fatou's lemma, and Lebesgue dominated convergence theorem already attest. There are no comparable theorems for the Riemann integral).

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Fubini's theorem (optional)

- In one dimension we have shown that the Lebesgue integral is connected to the Riemann integral. Now we will try to understand the connection in higher dimensions. To simplify the discussion we shall just study two-dimensional integrals, although the arguments we present here can easily be extended to higher dimensions.

- We shall study integrals of the form $\int_{\mathbf{R}^2} f$. Note that once we know how to integrate on $\mathbf{R}^2$, we can integrate on measurable subsets Ω of $\mathbf{R}^2$, since $\int_\Omega f$ can be rewritten as $\int_{\mathbf{R}^2} f\chi_\Omega$.

- Let $f(x,y)$ be a function of two variables. In principle, we have three different ways to integrate f on $\mathbf{R}^2$. First of all, we can use the two-dimensional Lebesgue integral, to obtain $\int_{\mathbf{R}^2} f$. Secondly, we can fix x and compute a one-dimensional integral in y, and then take that quantity and integrate in x, thus obtaining $\int_{\mathbf{R}}(\int_{\mathbf{R}} f(x,y)\ dy)\ dx$. Secondly, we could fix y and integrate in x, and then integrate in y, thus obtaining $\int_{\mathbf{R}}(\int_{\mathbf{R}} f(x,y)\ dx)\ dy$.

- Fortunately, if the function f is absolutely integrable on f, then all three integrals are equal:

- **Fubini's theorem.** Let $f : \mathbf{R}^2 \to \mathbf{R}^*$ be an absolutely integrable function. Then there exists absolutely integrable functions $F : \mathbf{R} \to \mathbf{R}$

and $G : \mathbf{R} \to \mathbf{R}$ such that for almost every x, $f(x, y)$ is absolutely integrable in y with

$$F(x) = \int_{\mathbf{R}} f(x, y)\ dy,$$

and for almost every y, $f(x, y)$ is absolutely integrable in x with

$$G(y) = \int_{\mathbf{R}} f(x, y)\ dx.$$

Finally, we have

$$\int_{\mathbf{R}} F(x)\ dx = \int_{\mathbf{R}^2} f = \int_{\mathbf{R}} G(y)\ dy.$$

- Very roughly speaking, Fubini's theorem says that

$$\int_{\mathbf{R}} (\int_{\mathbf{R}} f(x, y)\ dy)\ dx = \int_{\mathbf{R}^2} f = \int_{\mathbf{R}} (\int_{\mathbf{R}} f(x, y)\ dx)\ dy.$$

- This allows us to compute two-dimensional integrals by splitting them into two one-dimensional integrals. The reason why we do not write Fubini's theorem this way, though, is that it is possible that the integral $\int_{\mathbf{R}} f(x, y)\ dy$ does not actually exist for every x, and similarly $\int_{\mathbf{R}} f(x, y)\ dx$ does not exist for every y; Fubini's theorem only asserts that these integrals only exist for *almost every* x and y. For instance, if $f(x, y)$ is the function which equals 1 when $x > 0$ and $y = 0$, equals -1 when $x < 0$ and $y = 0$, and is zero otherwise, then f is absolutely integrable on $\mathbf{R}^2$ and $\int_{\mathbf{R}^2} f = 0$ (since f equals zero almost everywhere in $\mathbf{R}^2$), but $\int_{\mathbf{R}} f(x, y)\ dy$ is not absolutely integrable when $x = 0$ (though it is absolutely integrable for every other x).

- The proof of Fubini's theorem is quite complicated and we will only give a sketch here. We begin with a series of reductions.

- Roughly speaking (ignoring issues relating to sets of measure zero), we have to show that

$$\int_{\mathbf{R}} (\int_{\mathbf{R}} f(x, y)\ dy)\ dx = \int_{\mathbf{R}^2} f$$

together with a similar equality with x and y reversed. We shall just prove the above equality, as the other one is very similar.

- First of all, it suffices to prove the theorem for non-negative functions, since the general case then follows by writing a general function f as a difference $f^+ - f^-$ of two non-negative functions, and applying Fubini's theorem to f^+ and f^- separately (and using Proposition 7(ab)). Thus we will henceforth assume that f is non-negative.

- Next, it suffices to prove the theorem for non-negative functions f supported on a bounded set such as $[-N, N] \times [-N, N]$ for some positive integer N. Indeed, once one obtains Fubini's theorem for such functions, one can then write a general function f as the supremum of such compactly supported functions as

$$f = \sup_{N>0} f\chi_{[-N,N]\times[-N,N]},$$

apply Fubini's theorem to each function $f\chi_{[-N,N]\times[-N,N]}$ separately, and then take suprema using the monotone convergence theorem. Thus we will henceforth assume that f is supported on $[-N, N] \times [-N, N]$.

- By another similar argument, it suffices to prove the theorem for non-negative simple functions supported on $[-N, N] \times [-N, N]$, since one can use Lemma 23 from last week's notes to arise f as the supremum of simple functions (which must also be supported on $[-N, N]$), apply Fubini's theorem to each simple function, and then take suprema using the monotone convergence theorem. Thus we may assume that f is a non-negative simple function supported on $[-N, N] \times [-N, N]$.

- Next, we see that it suffices to prove the theorem for characteristic functions supported in $[-N, N] \times [-N, N]$. This is because every simple function is a linear combination of characteristic functions, and so we can deduce Fubini's theorem for simple functions from Fubini's theorem for characteristic functions. Thus we may take $f = \chi_E$ for some measurable $E \subseteq [-N, N] \times [-N, N]$. Our task is then to show (ignoring sets of measure zero) that

$$\int_{[-N,N]} (\int_{[-N,N]} \chi_E(x, y)\ dy)\ dx = m(E).$$

- It will suffice to show the upper Lebesgue integral estimate

$$\overline{\int}_{[-N,N]} (\overline{\int}_{[-N,N]} \chi_E(x,y)\ dy)\ dx \leq m(E).$$

We will prove this estimate later. Once we show this for every set E, we may substitute E with $[-N,N] \times [-N,N] \backslash E$ and obtain

$$\overline{\int}_{[-N,N]} (\overline{\int}_{[-N,N]} (1 - \chi_E(x,y))\ dy)\ dx \leq 4N^2 - m(E).$$

But the left-hand side is equal to

$$\overline{\int}_{[-N,N]} (2N - \underline{\int}_{[-N,N]} \chi_E(x,y)\ dy)\ dx$$

which is in turn equal to

$$4N^2 - \underline{\int}_{[-N,N]} (\underline{\int}_{[-N,N]} \chi_E(x,y)\ dy)\ dx$$

and thus we have

$$\underline{\int}_{[-N,N]} (\underline{\int}_{[-N,N]} \chi_E(x,y)\ dy)\ dx \geq m(E).$$

In particular we have

$$\underline{\int}_{[-N,N]} (\overline{\int}_{[-N,N]} \chi_E(x,y)\ dy)\ dx \geq m(E)$$

and hence by Lemma 8 we see that $\overline{\int}_{[-N,N]} \chi_E(x,y)\ dy$ is absolutely integrable and

$$\int_{[-N,N]} (\overline{\int}_{[-N,N]} \chi_E(x,y)\ dy)\ dx = m(E).$$

A similar argument shows that

$$\int_{[-N,N]} (\underline{\int}_{[-N,N]} \chi_E(x,y)\ dy)\ dx = m(E)$$

and hence

$$\int_{[-N,N]} (\overline{\int}_{[-N,N]} \chi_E(x,y)\ dy - \underline{\int}_{[-N,N]} \chi_E(x,y))\ dx = 0.$$

Thus by Proposition 3(a) we have

$$\underline{\int}_{[-N,N]} \chi_E(x,y)\ dy = \overline{\int}_{[-N,N]} \chi_E(x,y)\ dy$$

for almost every $x \in [-N,N]$. Thus $\chi_E(x,y)$ is absolutely integrable in y for almost every x, and $\int_{[-N,N]} \chi_E(x,y)$ is thus equal (almost everywhere) to a function $F(x)$ such that

$$\int_{[-N,N]} F(x)\ dx = m(E)$$

as desired.

- It remains to prove the bound

$$\overline{\int}_{[-N,N]} (\overline{\int}_{[-N,N]} \chi_E(x,y)\ dy)\ dx \leq m(E).$$

Let $\varepsilon > 0$ be arbitrary. Since $m(E)$ is the same as the outer measure $m^*(E)$, we know that there exists an at most countable collection $(B_j)_{j \in J}$ of boxes such that $E \subseteq \bigcup_{j \in J} B_j$ and

$$\sum_{j \in J} m(B_j) \leq m(E) + \varepsilon.$$

Each box B_j can be written as $B_j = I_j \times I'_j$ for some intervals I_j and I'_j. Observe that

$$m(B_j) = |I_j||I'_j| = \int_{I_j} |I'_j|\ dx = \int_{I_j} (\int_{I'_j}\ dy) dx$$

$$= \int_{[-N,N]} (\int_{[-N,N]} \chi_{I_j \times I'_j}(x,y)\ dx) dy = \int_{[-N,N]} (\int_{[-N,N]} \chi_{B_j}(x,y)\ dx) dy.$$

Adding this over all $j \in J$ (using Corollary 5) we obtain

$$\sum_{j\in J} m(B_j) = \int_{[-N,N]} (\int_{[-N,N]} \sum_{j\in J} \chi_{B_j}(x,y)\ dx)dy.$$

In particular we have

$$\overline{\int}_{[-N,N]} (\overline{\int}_{[-N,N]} \sum_{j\in J} \chi_{B_j}(x,y)\ dx)dy \leq m(E) + \varepsilon.$$

But $\sum_{j\in J} \chi_{B_j}$ majorizes χ_E (why?) and thus

$$\overline{\int}_{[-N,N]} (\overline{\int}_{[-N,N]} \chi_E(x,y)\ dx)dy \leq m(E) + \varepsilon.$$

But ε is arbitrary, and so we have

$$\overline{\int}_{[-N,N]} (\overline{\int}_{[-N,N]} \chi_E(x,y)\ dy)\ dx \leq m(E).$$

as desired. This completes the proof of Fubini's theorem.