Math 131BH - Week 1
Textbook pages covered: 30-36, 48-55

- Overview of course
- Metric spaces
- Convergence in metric spaces
- Open and closed sets

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Overview of course

- This course is a continuation of Math 131AH, Honors real analysis. In that course we rigorously studied the foundations of calculus of one variable: the real number system, limits of sequences and series, continuity, differentiation, and the Riemann integral. In this course we shall continue these studies, but now we will be working in more general contexts. For instance, we will be studying convergence, limits and continuity, not just in the real numbers as in 131AH, but in more general settings known as *metric spaces*. In particular, we will be able to use these concepts in several dimensions, which then leads to notions of derivatives and integrals in several variables. The notion of convergence of series was studied in 131AH; we shall return to this notion again, but now we will be summing *functions* instead of *numbers*. In particular we will study two important classes of series of functions: *power series* and *Fourier series*. This in particular allows us to rigorously introduce many familiar functions, such as exp, sin, log, etc., all of which play a basic role in modern mathematics.

- Many of these notions were already covered in Math 33B, but we will be reviewing them in much greater depth than in that course. There is also some overlap with other upper-division courses such as Math 121 (Introduction to Topology) and Math 133 (Introduction to Fourier Analysis).

- In the last three weeks of the course, we will begin addressing the question of what a volume of a set in several dimensions (e.g. in $\mathbf{R}^3$) really means; it turns out that this is actually a rather subtle concept, and the Riemann integral begins to run into difficulties when trying to satisfactorily answer this question. An example: consider the set $\{(x, y, z) \in \mathbf{R}^3 : 0 \leq x, y, z \leq 1; x, y, z \text{ irrational } \}$, i.e. all the points in the unit cube with irrational co-ordinates. What is the volume of this set? We shall develop a more powerful version of the Riemann integral, known as the *Lebesgue integral*, which gives a satisfactory answer to the question of how to integrate functions, and how to compute volumes of sets such as the one described above.

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Metric spaces

- Recall from Math 131AH that we know what it means for a sequence $(x_n)_{n=m}^\infty$ of real numbers to converge to another real number x; this means that for every $\varepsilon > 0$, there exists an $N \geq m$ such that $|x - x_n| \leq \varepsilon$ for all $n \geq N$. When this is the case, we write $\lim_{n\to\infty} x_n = x$.

- Intuitively, when a sequence $(x_n)_{n=m}^\infty$ converges to a limit x, this means that somehow the elements x_n of that sequence will eventually be as close to x as one pleases. One way to phrase this more precisely is to introduce the *distance function* $d(x, y)$ between two real numbers by $d(x, y) := |x - y|$. (Thus for instance $d(3, 5) = 2$, $d(5, 3) = 2$, and $d(3, 3) = 0$). Then we have

- **Lemma 1.** Let $(x_n)_{n=m}^\infty$ be a sequence of real numbers, and let x be another real number. Then $(x_n)_{n=m}^\infty$ converges to x if and only if $\lim_{n\to\infty} d(x_n, x) = 0$.

- **Proof.** See Week 1 homework. □

- One would now like to generalize this notion of convergence, so that one can take limits not just of sequences of real numbers, but also sequences of complex numbers, or sequences of vectors, or sequences of matrices, or sequences of functions, even sequences of sequences. One way to do this is to redefine the notion of convergence each time we deal with a new type of object. As you can guess, this will quickly get tedious. A more efficient way is to work *abstractly*, defining a very general class of spaces - which includes such standard spaces as the real numbers, complex numbers, vectors, etc. - and define the notion of convergence on this entire class of spaces at once. (A space is just the set of all objects of a certain type - the space of all real numbers, the space of all 3×3 matrices, etc. Mathematically, there is not much distinction between a space and a set, except that spaces tend to have much more structure than what a random set would have. For instance, the space of real numbers comes with operations such as addition and multiplication, while a general set would not).

- It turns out that there are two very useful classes of spaces which do the job. The first class is that of *metric spaces*, which we will study here. There is a more general class of spaces, called *topological spaces*, which are also very important, but we will not deal with these in this course; you will have to go to Math 121, Introduction to Topology, to learn more about them.

- Roughly speaking, a metric space is any space X which has a concept of *distance* $d(x, y)$ - and this distance should behave in a reasonable manner. More precisely, we have

- **Definition** A *metric space* (X, d) is a space X of objects (called *points*), together with a *distance function* or *metric* $d : X \times X \to [0, \infty)$, which associates to each pair x, y of points in X a non-negative real number $d(x, y) \geq 0$. Furthermore, the metric must satisfy the following four axioms:

- (i) For any $x \in X$, we have $d(x, x) = 0$.

- (ii) (Positivity) For any *distinct* $x, y \in X$, we have $d(x, y) > 0$.

- (iii) (Symmetry) For any $x, y \in X$, we have $d(x, y) = d(y, x)$.

- (iv) (Triangle inequality) For any $x, y, z \in X$, we have $d(x, z) \leq d(x, y) + d(y, z)$.

- In many cases it will be clear what the metric d is, and we shall abbreviate (X, d) as just X.

- Remark. The conditions (i) and (ii) can be rephrased as follows: for any $x, y \in X$ we have $d(x, y) = 0$ if and only if $x = y$. (Why is this equivalent to (i) and (ii)?).

- **Example. (The real line)** Let $\mathbf{R}$ be the real numbers, and let $d : \mathbf{R} \times \mathbf{R} \to [0, \infty)$ be the metric $d(x, y) := |x - y|$ mentioned earlier. Then $(\mathbf{R}, d)$ is a metric space. (We leave it to the reader to verify all the properties. The triangle inequality is the trickiest. See also the Week 2 notes from my Math 131AH course.)

- **Example. (The integers)** Let $\mathbf{Z}$ be the integers, and let $d|_{\mathbf{Z}\times\mathbf{Z}} : \mathbf{Z} \times \mathbf{Z} \to [0, \infty)$ be the metric function $d|_{\mathbf{Z}\times\mathbf{Z}}(x, y) := |x - y|$; i.e. $d|_{\mathbf{Z}\times\mathbf{Z}}$ is the restriction of the real line metric $d : \mathbf{R} \times \mathbf{R} \to [0, \infty)$ to $\mathbf{Z} \times \mathbf{Z}$. Thus for instance $d(1, 4) = 3$ and $d(6, 2) = 4$. Then $(\mathbf{Z}, d|_{\mathbf{Z}\times\mathbf{Z}})$ is also a metric space; this is what we call a *subspace* of the larger metric space $(\mathbf{R}, d)$. (More generally, if (X, d) is a metric space, and Y is a subset of X, then $(Y, d|_{Y\times Y})$ is also a metric space, and is called a *subspace* of (X, d), or the *restriction* of the metric space (X, d) to Y, or the metric space on Y *induced* by (X, d).) Note that if the larger space (X, d) obeys the axioms (i)-(iv), then the subspace $(Y, d|_{Y\times Y})$ will automatically obey these axioms also. (Why?)

- **Example. (Euclidean spaces)** Let $n \geq 1$ be a natural number, and let $\mathbf{R}^n$ be the space of n-tuples of real numbers:

$$\mathbf{R}^n = \{(x_1, x_2, \ldots, x_n) : x_1, \ldots, x_n \in \mathbf{R}\}.$$

(Equivalently, one can define $\mathbf{R}^n$ inductively using Cartesian products, as $\mathbf{R}^1 := \mathbf{R}$, and $\mathbf{R}^{n+1} := \mathbf{R}^n \times \mathbf{R}$ for all $n \geq 1$). We define the *Euclidean metric* (also called the l^2 *metric*) $d_{l^2} : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ by

$$d_{l^2}((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)) := \sqrt{(x_1 - y_1)^2 + \ldots + (x_n - y_n)^2} = (\sum_{i=1}^{n} (x_i - y_i)^2)^{1/2}.$$

Thus for instance, if $n = 2$, then $d_{l^2}((1, 6), (4, 2)) = \sqrt{3^2 + 4^2} = 5$. This metric corresponds to the geometric distance between the two points $(x_1, x_2, \ldots, x_n)$, $(y_1, y_2, \ldots, y_n)$ as given by Pythagoras's theorem. (We remark however that while geometry does give some very important examples of metric spaces, it is possible to have metric spaces which have no obvious geometry whatsoever. Some examples are given below). The verification that $(\mathbf{R}^n, d)$ is indeed a metric space can be seen geometrically (for instance, the triangle inequality now asserts that the length of one side of a triangle is always less than or equal to the sum of the lengths of the other two sides), but can also be proven algebraically (see exercises).

- **Example. (Taxicab metric)** Again let $n \geq 1$, and let $\mathbf{R}^n$ be as before. But now we use a different metric d_{l^1}, the so-called *taxicab metric* (or l^1 *metric*), defined by

$$d_{l^1}((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)) := |x_1 - y_1| + \ldots + |x_n - y_n| = \sum_{i=1}^{n} |x_i - y_i|.$$

Thus for instance, if $n = 2$, then $d_{l^1}((1, 6), (4, 2)) = 5 + 2 = 7$. This metric is called the taxi-cab metric, because it models the distance a taxi-cab would have to traverse to get from one point to another if the cab was only allowed to move in cardinal directions (north, south, east, west) and not diagonally. As such it is always at least as large as the Euclidean metric, which measures distance "as the crow flies", as it were. (Can you see algebraically why d_{l^1} is always greater than or equal to d_{l^2}? Try squaring both sides). We claim that the space $(\mathbf{R}^n, d_{l^1})$ is also a metric space. We shall only verify one of the properties, namely the triangle inequality (iv); we leave the other three properties to the reader. Let $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$, and $z = (z_1, \ldots, z_n)$ be three points in $\mathbf{R}^n$. We have to show that

$$d_{l^1}(x, z) \leq d_{l^1}(x, y) + d_{l^1}(y, z).$$

We expand this as

$$\sum_{i=1}^{n} |x_i - z_i| \leq \sum_{i=1}^{n} |x_i - y_i| + \sum_{i=1}^{n} |y_i - z_i|.$$

But by the triangle inequality for $\mathbf{R}$, we have $|x_i - z_i| \leq |x_i - y_i| + |y_i - z_i|$ for all $i = 1, \ldots, n$, and the claim follows.

- The taxi-cab metric is useful in several places, for instance in the theory of error correcting codes. A string of n binary digits can be thought of as an element of $\mathbf{R}^n$, for instance the binary string 10010 can be thought of as the point $(1,0,0,1,0)$ in $\mathbf{R}^5$. The taxi-cab distance between two binary strings is then the number of bits in the two strings which do not match, for instance $d_{l^1}(10010, 10101) = 3$. The goal of error-correcting codes is to encode each piece of information (e.g. a letter of the alphabet) as a binary string in such a way that all the binary strings are as far away in the taxicab metric from each other as possible; this minimizes the chance that any distortion of the bits due to random noise can accidentally change one of the coded binary strings to another, and also maximizes the chance that any such distortion can be detected and correctly repaired.

- There are in fact l^p metrics for every positive real number p, and there is also an l^∞ metric, but we will not discuss those in this course.

- **Example. (Discrete metric)** Let X be an arbitrary set (finite or infinite), and define the *discrete metric* d_{disc} by setting $d_{disc}(x,y) := 0$ when $x = y$, and $d_{disc}(x,y) := 1$ when $x \neq y$. Thus, in this metric, all points are equally far apart. One can easily verify that (X, d_{disc}) is a metric space for any set X (why?).

- **Example. (Geodesics)** Let X be the sphere $\{(x,y,z) \in \mathbf{R}^3 : x^2 + y^2 + z^2 = 1\}$, and let $d((x,y,z),(x',y',z'))$ be the length of the shortest curve in X which starts at (x,y,z) and ends at (x',y',z'). (This curve turns out to be an arc of a great circle; we will not prove this here, as it requires *calculus of variations*, which is beyond the scope of this course). This makes X into a metric space; the reader should be able to verify (without using any geometry of the sphere) that the triangle inequality is more or less automatic from the definition.

- **Example. (Shortest paths)** Examples of metric spaces occur all the time in real life. For instance, X could be all the computers currently connected to the internet, and $d(x,y)$ is the shortest number of connections it would take for a packet to travel from computer x to computer y; for instance, if x and y are not directly connected, but are both connected to z, then $d(x,y) = 2$. Assuming that all computers in the internet can ultimately be connected to all other computers (so that $d(x, y)$ is always finite), then (X, d) is a metric space (why?). Games such as "six degrees of separation" are also taking place in a similar metric space (what is the space, and what is the metric, in this case?). Or, X could be Los Angeles, and $d(x, y)$ could be the shortest time it takes to drive from x to y (although this space might not satisfy axiom (iii) in real life!).

- Now that we have metric spaces, we can define convergence in these spaces.

- **Definition** Let m be an integer, (X, d) be a metric space and let $(x^{(n)})_{n=m}^{\infty}$ be a sequence of points in X (i.e. for every natural number $n \geq m$, we assume that $x^{(n)}$ is an element of X). Let x be a point in X. We say that $(x^{(n)})_{n=m}^{\infty}$ *converges to x with respect to the metric d*, if and only if the limit $\lim_{n\to\infty} d(x^{(n)}, x)$ exists and is equal to 0. In other words, $(x^{(n)})_{n=m}^{\infty}$ converges to x with respect to d if and only if for every $\varepsilon > 0$, there exists an $N \geq m$ such that $d(x^{(n)}, x) \leq \varepsilon$ for all $n \geq N$. (Why are these two definitions equivalent?)

- Note that this definition is consistent with Lemma 1, in that it generalizes our existing notion of convergence of sequences of real numbers. In many cases, it is obvious what the metric d is, and so we shall often just say "$(x^{(n)})_{n=m}^{\infty}$ converges to x" instead of "$(x^{(n)})_{n=m}^{\infty}$ converges to x with respect to the metric d" when there is no chance of confusion. We also sometimes write "$x^{(n)} \to x$ as $n \to \infty$" instead.

- Of course, there is nothing special about the subscript n; it is a dummy variable. Saying that $(x^{(n)})_{n=m}^{\infty}$ converges to x is exactly the same statement as saying that $(x^{(k)})_{k=m}^{\infty}$ converges to x, for example; and sometimes it is convenient to change subscripts, for instance if the variable n is already being used for some other purpose. Similarly, it is not necessary for the sequence $x^{(n)}$ to be denoted using the superscript (n); the above definition is also valid for sequences x_n, or functions $f(n)$, or indeed of any expression which depends on n and takes values in X. Finally, the starting point m of the sequence is unimportant for the purposes of taking limits; if $(x^{(n)})_{n=m}^{\infty}$ converges to x, then $(x^{(n)})_{n=m'}^{\infty}$ also converges to x for any $m' \geq m$. (Why?)

- **Example** We work in the Euclidean space $\mathbf{R}^2$ with the standard Euclidean metric d_{l^2}. Let $(x^{(n)})_{n=1}^\infty$ denote the sequence $x^{(n)} := (1/n, 1/n)$ in $\mathbf{R}^2$, i.e. we are considering the sequence $(1,1), (1/2,1/2), (1/3,1/3), \ldots$. Then this sequence converges to $(0,0)$ with respect to the Euclidean metric d_{l^2}, since

$$\lim_{n\to\infty} d_{l^2}(x^{(n)}, (0,0)) = \lim_{n\to\infty} \sqrt{\frac{1}{n^2} + \frac{1}{n^2}} = \lim_{n\to\infty} \frac{\sqrt{2}}{n} = 0.$$

The sequence $(x^{(n)})_{n=1}^\infty$ also converges to $(0,0)$ with respect to the taxicab metric d_{l^1}, since

$$\lim_{n\to\infty} d_{l^1}(x^{(n)}, (0,0)) = \lim_{n\to\infty} \frac{1}{n} + \frac{1}{n} = \lim_{n\to\infty} \frac{2}{n} = 0.$$

However, the sequence $(x^{(n)})_{n=1}^\infty$ does *not* converge to $(0,0)$ in the discrete metric d_{disc}, since

$$\lim_{n\to\infty} d_{disc}(x^{(n)}, (0,0)) = \lim_{n\to\infty} 1 = 1 \neq 0.$$

Thus the convergence of a sequence can depend on what metric one uses. (Perhaps one can think of a real-life example comparing, say, the automobile metric and the pedestrian metric in Los Angeles).

- In the case of the above three metrics - Euclidean, taxicab, and discrete - it is in fact rather easy to test for convergence.

- **Proposition 2.** Let $\mathbf{R}^n$ be a Euclidean space, and let $(x^{(k)})_{k=m}^\infty$ be a sequence of points in $\mathbf{R}^n$. We write $x^{(k)} = (x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)})$, i.e. for $j = 1, 2, \ldots, n$, $x_j^{(k)} \in \mathbf{R}$ is the j^{th} co-ordinate of $x^{(k)} \in \mathbf{R}^n$. Let $x = (x_1, \ldots, x_n)$ be a point in $\mathbf{R}^n$. Then the following three statements are equivalent:

- (a) $(x^{(k)})_{k=m}^\infty$ converges to x with respect to the Euclidean metric d_{l^2}.

- (b) $(x^{(k)})_{k=m}^\infty$ converges to x with respect to the taxicab metric d_{l^1}.

- (c) For every $1 \leq j \leq n$, the sequence $(x_j^{(k)})_{k=m}^\infty$ converges to x_j. (Notice that this is a sequence of real numbers, not of points in $\mathbf{R}^n$).

- **Proof.** See Week 1 homework. □

- In other words, a sequence converges in the Euclidean or taxicab metric if and only if each of its components converges individually. Because of the equivalence of (a) and (b), we say that the Euclidean metric and the taxicab metric on $\mathbf{R}^n$ are *equivalent*. (There are infinite-dimensional analogues of the Euclidean and taxicab metrics which are *not* equivalent, but that is a matter beyond the scope of this course).

- For the discrete metric, convergence is much harder: the sequence must be eventually constant in order to converge.

- **Proposition 3.** Let X be any set, and let d_{disc} be the discrete metric on X. Let $(x^{(n)})_{n=m}^\infty$ be a sequence of points in X, and let x be a point in X. Then $(x^{(n)})_{n=m}^\infty$ converges to x with respect to the discrete metric d_{disc} if and only if there exists an $N \geq m$ such that $x^{(n)} = x$ for all $n \geq N$.

- **Proof.** See Week 1 homework. □

- We now prove a basic fact about converging sequences; they can only converge to at most one point at a time.

- **Proposition 4.** Let (X, d) be a metric space, and let $(x^{(n)})_{n=m}^\infty$ be a sequence in X. Suppose that there are two points $x, x' \in X$ such that $(x^{(n)})_{n=m}^\infty$ converges to x with respect to d, and $(x^{(n)})_{n=m}^\infty$ also converges to x' with respect to d. Then we have $x = x'$.

- **Proof.** See Week 1 homework. □

- Because of the above Proposition, it is safe to introduce the following notation: if $(x^{(n)})_{n=m}^\infty$ converges to x in the metric d, then we write $d - \lim_{n\to\infty} x^{(n)} = x$, or simply $\lim_{n\to\infty} x^{(n)} = x$ when there is no confusion as to what d is. For instance, in the example of $(\frac{1}{n}, \frac{1}{n})$, we have
$$d_{l^2} - \lim_{n\to\infty} (\frac{1}{n}, \frac{1}{n}) = d_{l^1} - \lim_{n\to\infty} (\frac{1}{n}, \frac{1}{n}) = (0, 0),$$
but $d_{disc} - \lim_{n\to\infty} (\frac{1}{n}, \frac{1}{n})$ is undefined. Thus the meaning of $d - \lim_{n\to\infty} x^{(n)}$ can depend on what d is; however Proposition 4 assures us that once d

is fixed, there can be at most one value of $d - \lim_{n\to\infty} x^{(n)}$. (Of course, it is still possible that this limit does not exist; some sequences are not convergent).

- It is even possible for a sequence to converge to one point using one metric, and another point using a different metric, although such examples are usually quite artificial. For instance, let $X := [0,1]$, the closed interval from 0 to 1. Using the usual metric d, we have $d - \lim_{n\to\infty} \frac{1}{n} = 0$. But now suppose we "swap" the points 0 and 1 in the following manner. Let $f : [0,1] \to [0,1]$ be the function defined by $f(0) := 1$, $f(1) := 0$, and $f(x) := x$ for all $x \in (0,1)$, and then define $d'(x,y) := d(f(x), f(y))$. Then (X, d') is still a metric space (why?), but now $d' - \lim_{n\to\infty} \frac{1}{n} = 1$. Thus changing the metric on a space can greatly affect the nature of convergence (also called the *topology*) on that space.

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Some point-set topology of metric spaces

- We have now defined the operation of convergence on metric spaces. We now define a couple other related notions, including that of open set, closed set, interior, exterior, boundary, and adherent point. The study of these notions (and a few others, some of which we will discuss next week) is known as *point-set topology*, although we will only skim the theory of topology, leaving the finer detail to Math 121.

- We first need the notion of a *metric ball*, or more simply a *ball*.

- **Definition** Let (X, d) be a metric space, let x_0 be a point in X, and let $r > 0$. We define the *ball* $B_{(X,d)}(x_0, r)$ in X, centered at x_0, and with radius r, in the metric d, to be the set

$$B_{(X,d)}(x_0, r) := \{x \in X : d(x, x_0) < r\}.$$

When it is clear what the metric space (X, d) is, we shall abbreviate $B_{(X,d)}(x_0, r)$ as just $B(x_0, r)$.

- **Example.** In $\mathbf{R}^2$ with the Euclidean metric d_{l^2}, the ball $B_{(\mathbf{R}^2, d_{l^2})}((0,0), 1)$ is the open disc

$$B_{(\mathbf{R}^2, d_{l^2})}((0,0), 1) = \{(x, y) \in \mathbf{R}^2 : x^2 + y^2 < 1\}.$$

However, if one uses the taxi-cab metric d_{l^1} instead, then we obtain a diamond:

$$B_{(\mathbf{R}^2, d_{l^1})}((0,0),1) = \{(x,y) \in \mathbf{R}^2 : |x| + |y| < 1\}.$$

If we use the discrete metric, the ball is now reduced to a single point:

$$B_{(\mathbf{R}^2, d_{disc})}((0,0),1) = \{(0,0)\},$$

although if one increases the radius to be larger than 1, then the ball now encompasses all of $\mathbf{R}^2$. (Why?)

- **Example.** In $\mathbf{R}$ with the usual metric d, the ball $B_{(\mathbf{R},d)}(5,2)$ is the open interval $(3,7)$.

- Note that the smaller the radius r, the smaller the ball $B(x_0, r)$. However, $B(x_0, r)$ always contains at least one point, namely the center x_0, as long as r stays positive, thanks to axiom (i). (We don't consider balls of zero radius or negative radius since they are rather boring, being just the empty set).

- Using metric balls, one can now take a set E a metric space, and classify three types of points in X: interior, exterior, and boundary points of E.

- **Definition** Let (X, d) be a metric space, let E be a subset of X, and let x_0 be a point in X. We say that x_0 is an *interior point of* E if there exists a radius $r > 0$ such that $B(x_0, r) \subseteq E$. We say that x_0 is an *exterior point of* E if there exists a radius $r > 0$ such that $B(x_0, r) \cap E = \emptyset$. We say that x_0 is a *boundary point of* E if it is neither an interior point nor an exterior point of E.

- The set of all interior points of E is called the *interior* of E and is sometimes denoted $int(E)$. The set of exterior points of E is called the *exterior* of E and is sometimes denoted $ext(E)$. The set of boundary points of E is called the *boundary* of E and is sometimes denoted ∂E.

- Note that if x_0 is an interior point of E, then x_0 must actually be an element of E, since balls $B(x_0, r)$ always contain their center x_0.

Conversely, if x_0 is an exterior point of E, then x_0 cannot be an element of E. In particular it is not possible for x_0 to simultaneously be an interior and an exterior point of E. If x_0 is a boundary point of E, then it could be an element of E, but it could also not lie in E; we give some examples below.

- **Example.** We work on the real line $\mathbf{R}$ with the standard metric d. Let E be the half-open interval $E = [1, 2)$. The point 1.5 is an interior point of E, since one can find a ball (for instance $B(1.5, 0.1)$) centered at 1.5 which lies in E. The point 3 is an exterior point of E, since one can find a ball (for instance $B(3, 0.1)$) centered at 3 which is disjoint from E. The points 1 and 2 however, are neither interior points nor exterior points of E, and are thus boundary points of E. Thus in this case $int(E) = (1, 2)$, $ext(E) = (-\infty, 1) \cup (2, \infty)$, and $\partial E = \{1, 2\}$. Note that in this case one of the boundary points is an element of E, while the other is not.

- **Example.** When we give a set X the discrete metric d_{disc}, and E is any subset of X, then every element of E is an interior point of E, every point not contained in E is an exterior point of E, and there are no boundary points. (Why?).

- **Definition** Let (X, d) be a metric space, let E be a subset of X, and let x_0 be a point in X. We say that x_0 is an *adherent point* of E if for every radius $r > 0$, the ball $B(x_0, r)$ has a non-empty intersection with E. The set of all adherent points of E is called the *closure* of E and is denoted $\overline{E}$.

- The following proposition links the notions of adherent point with interior and boundary point, and also to that of convergence.

- **Proposition 5.** Let (X, d) be a metric space, let E be a subset of X, and let x_0 be a point in X. Then the following statements are equivalent.

- (a) x_0 is an adherent point of E.

- (b) x_0 is either an interior point or a boundary point of E.

- (c) There exists a sequence $(x_n)_{n=1}^{\infty}$ in E which converges to x_0 with respect to the metric d.

- **Proof.** See Week 1 homework. $\square$

- As remarked earlier, the boundary of a set E may or may not lie in E. Depending on how the boundary is situated, we may call a set open, closed, or neither:

- **Definition** Let (X, d) be a metric space, and let E be a subset of X. We say that E is *closed* if it contains all of its boundary points, i.e. $\partial E \subseteq E$. We say that E is *open* if it contains none of its boundary points, i.e. $\partial E \cap E = \emptyset$. If E contains some of its boundary points but not others, then it is neither open nor closed.

- **Example.** We work in the real line $\mathbf{R}$ with the standard metric d. The set $(1, 2)$ does not contain either of its boundary points 1, 2 and is hence open. The set $[1, 2]$ contains both of its boundary points 1, 2 and is hence closed. The set $[1, 2)$ contains one of its boundary points 1, but does not contain the othe boundary point 2, but not the other, so is neither open nor closed.

- It is possible for a set to be simultaneously open and closed, if it has no boundary. For instance, in a metric space (X, d), the whole space X has no boundary (every point in X is an interior point - why?), and so X is both open and closed. The empty set $\emptyset$ also has no boundary (every point in X is an exterior point - why?), and so is both open and closed. In many cases these are the only sets that are simultaneously open and closed, but there are exceptions. For instance, using the discrete metric d_{disc}, *every* set is both open and closed! (why?)

- Now we list some more properties of open and closed sets.

- **Proposition 6** Let (X, d) be a metric space.

- (a) Let E be a subset of X. Then E is open if and only if $E = int(E)$. In other words, E is open if and only if for every $x \in E$, there exists an $r > 0$ such that $B(x, r) \subseteq E$.

- (b) Let E be a subset of X. Then E is closed if and only if E contains all its adherent points. In other words, E is closed if and only if for every convergent sequence $(x_n)_{n=m}^{\infty}$ in E, the limit $\lim_{n\to\infty} x_n$ of that sequence also lies in E.

- (c) For any $x_0 \in X$ and $r > 0$, then the ball $B(x_0, r)$ is an open set. The set $\{x \in X : d(x, x_0) \leq r\}$ is a closed set. (This set is sometimes called the *closed ball* of radius r centered at x_0).

- (d) Any singleton set $\{x_0\}$, where $x_0 \in X$, is automatically closed.

- (e) If E is a subset of X, then E is open if and only if $X \backslash E$ is closed. ($X \backslash E := \{x \in X : x \notin E\}$ is the complement of E in X).

- (f) If $E_1, \ldots, E_n$ are a finite collection of open sets in X, then $E_1 \cap E_2 \cap \ldots \cap E_n$ is also open. If $F_1, \ldots, F_n$ is a finite collection of closed sets in X, then $F_1 \cup F_2 \cup \ldots \cup F_n$ is also closed.

- (g) If $\{E_\alpha\}_{\alpha \in A}$ is a collection of open sets in X (where the index set A could be finite, countable, or uncountable), then the union $\bigcup_{\alpha \in A} E_\alpha := \{x \in X : x \in E_\alpha \text{ for some } \alpha \in A\}$ is also open. If $\{F_\alpha\}_{\alpha \in A}$ is a collection of closed sets in X, then the intersection $\bigcap_{\alpha \in A} F_\alpha := \{x \in X : x \in F_\alpha \text{ for all } \alpha \in A\}$ is also closed.

- (h) If E is any subset of X, then $int(E)$ is the largest open set which is contained in E; in other words, $int(E)$ is open, and given any other open set $V \subseteq E$, we have $V \subseteq int(E)$. Similarly $\overline{E}$ is the smallest closed set which contains E; in other words, $\overline{E}$ is closed, and given any other closed set $K \supset E$, $K \supset \overline{E}$.

- **Proof.** See Week 1 homework. □

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Relative topology

- When we defined notions such as open and closed sets, we mentioned that such concepts depended on the choice of metric one uses. For instance, on the real line $\mathbf{R}$, if one uses the usual metric $d(x, y) = |x-y|$, then the set $\{1\}$ is not open, however if instead one uses the discrete metric d_{disc}, then $\{1\}$ is now an open set (why?).

- However, it is not just the choice of metric which determines what is open and what is not - it is also the choice of *ambient space* X. Here are some examples.

- **Example** Consider the plane $\mathbf{R}^2$ with the Euclidean metric d_{l^2}. Inside the plane, we can find the x-axis $X := \{(x, 0) : x \in \mathbf{R}\}$. The metric d_{l^2} can be restricted to X, creating a subspace $(X, d_{l^2}|_{X \times X})$ of $(\mathbf{R}^2, d_{l^2})$. (This subspace is essentially the same as the real line $(\mathbf{R}, d)$ with the usual metric; the precise way of stating this is that $(X, d_{l^2}|_{X \times X})$ is *isometric* to $(\mathbf{R}, d)$. We will not pursue this concept further in this course, however). Now consider the set

  $$E := \{(x, 0) : -1 < x < 1\}$$

  which is both a subset of X and of $\mathbf{R}^2$. Viewed as a subset of $\mathbf{R}^2$, it is not open, because the point 0, for instance, lies in E but is not an interior point of E. (Any ball $B_{\mathbf{R}^2, d_{l^2}}(0, r)$ will contain at least one point that lies outside of the x-axis, and hence outside of E. On the other hand, if viewed as a subset of X, it is open; every point of E is an interior point of E *with respect to the metric space* $(X, d_{l^2}|_{X \times X})$. For instance, the point 0 is now an interior point of E, because the ball $B_{X, d_{l^2}|_{X \times X}}(0, 1)$ is contained in E (in fact, in this case it *is* E).

- **Example** Consider the real line $\mathbf{R}$ with the standard metric d, and let X be the interval $X := (-1, 1)$ contained inside $\mathbf{R}$; we can then restrict the metric d to X, creating a subspace $(X, d|_{X \times X})$ of $(\mathbf{R}, d)$. Now consider the set $[0, 1)$. This set is not closed in $\mathbf{R}$, because the point 1 is adherent to $[0, 1)$ but is not contained in $[0, 1)$. However, when considered as a subset of X, the set $[0, 1)$ now becomes closed; the point 1 is not an element of X and so is no longer considered an adherent point of $[0, 1)$, and so now $[0, 1)$ contains all of its adherent points.

- To clarify this distinction, we make a definition.

- **Definition.** Let (X, d) be a metric space, let Y be a subset of X, and let E be a subset of Y. We say that E is *relatively open with respect to* Y if it is open in the metric space $(Y, d|_{Y \times Y})$. Similarly, we say that E

is *relatively closed with respect to* Y if it is closed in the metric space $(Y, d|_{Y\times Y})$.

- The relationship between open (or closed) sets in X, and relatively open (or relatively closed) sets in Y, is the following.

- **Proposition 7** Let (X, d) be a metric space, let Y be a subset of X, and let E be a subset of Y.

- (a) E is relatively open with respect to Y if and only if $E = V \cap Y$ for some set $V \subseteq X$ which is open in X.

- (b) E is relatively closed with respect to Y if and only if $E = K \cap Y$ for some set $K \subseteq X$ which is closed in X.

- **Proof.** We just prove (a), and leave (b) to the exercises. First suppose that E is relatively open with respect to Y. Then, E is open in the metric space $(Y, d|_{Y\times Y})$. Thus, for every $x \in E$, there exists a radius $r > 0$ such that the ball $B_{(Y,d|_{Y\times Y})}(x, r)$ is contained in E. This radius r depends on x; to emphasize this we write r_x instead of r, thus for every $x \in E$ the ball $B_{(Y,d|_{Y\times Y})}(x, r)$ is contained in E.

- Now consider the set
$$V := \bigcup_{x\in E} B_{(X,d)}(x, r_x).$$
This is a subset of X. By Proposition 6(cg), V is open. Now we prove that $E = V \cap Y$. Certainly any point x in E lies in $V \cap Y$, since it lies in Y and it also lies in $B_{(X,d)}(x, r_x)$, and hence in V. Now suppose that y is a point in $V \cap Y$. Then $y \in V$, which implies that there exists an $x \in E$ such that $y \in B_{(X,d)}(x, r_x)$. But since y is also in Y, this implies that $y \in B_{(Y,d|_{Y\times Y})}(x, r_x)$. But by definition of r_x, this means that $y \in E$, as desired. Thus we have found an open set V for which $E = V \cap Y$ as desired.

- Now we do the converse. Suppose that $E = V \cap Y$ for some open set V; we have to show that E is relatively open with respect to Y. Let x be any point in E; we have to show that x is an interior point of E in the metric space $(Y, d|_{Y\times Y})$. Since $x \in E$, we know $x \in V$.

Since V is open in X, we know that there is a radius $r > 0$ such that $B_{(X,d)}(x,r)$ is contained in V. Since $E = V \cap Y$, this means that $B_{(X,d)}(x,r) \cap Y$ is contained in E. But $B_{(X,d)}(x,r) \cap y$ is exactly the same as $B_{(Y,d|_{Y\times Y})}(x,r)$ (why?), and so $B_{(Y,d|_{Y\times Y})}(x,r)$ is contained in E. Thus x is an interior point of E in the metric space $(Y, d|_{Y\times Y})$, as desired.

- The corresponding claims for closed sets are left to the exercises. $\square$