Math 131AH - Week 6
Textbook pages: 51-52,83-89.
Topics covered:

- Subsequences
- The Bolzano-Weierstrass theorem
- Functions on the real line
- Limiting values of functions
- Continuous functions

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Subsequences

- In last week's notes, we discussed all kinds of sequences $(a_n)_{n=1}^\infty$ of real numbers. Some sequences were convergent, but others were not. For instance, the sequence

  $$1.1, 0.1, 1.01, 0.01, 1.001, 0.001, 1.0001, \ldots$$

  has two limit points at 0 and 1 (which are incidentally also the lim inf and lim sup respectively), but is not actually convergent (since the lim sup and lim inf are not equal). However, while this sequence is not convergent, it does appear to contain convergent components; it seems to be a mixture of two convergent subsequences, namely

  $$1.1, 1.01, 1.001, \ldots$$

  and

  $$0.1, 0.01, 0.001, \ldots.$$

  To make this notion more precise, we need a notion of subsequence.

- **Definition.** Let $(a_n)_{n=0}^\infty$ and $(b_n)_{n=0}^\infty$ be sequences of real numbers. We say that $(b_n)_{n=0}^\infty$ is a *subsequence* of $(a_n)_{n=0}^\infty$ iff there exists a function $f : \mathbf{N} \to \mathbf{N}$ which is strictly increasing (i.e. $f(n++) > f(n)$ for all $n \in \mathbf{N}$) such that

  $$b_n = a_{f(n)} \text{ for all } n \in \mathbf{N}.$$

- For instance, the sequence $(a_{2n})_{n=0}^\infty$ is a subsequence of $(a_n)_{n=0}^\infty$, since the function $f : \mathbf{N} \to \mathbf{N}$ defined by $f(n) := 2n$ is a strictly increasing function from $\mathbf{N}$ to $\mathbf{N}$. (Note that we do not assume this function to be bijective, although it is necessarily injective (why?)). More informally, the sequence

$$a_0, a_2, a_4, a_6, \ldots$$

is a subsequence of

$$a_0, a_1, a_2, a_3, a_4, \ldots.$$

- Thus, for instance, the two sequences

$$1.1, 1.01, 1.001, \ldots$$

and

$$0.1, 0.01, 0.001, \ldots$$

mentioned earlier are both subsequences of

$$1.1, 0.1, 1.01, 0.01, 1.001, 1.0001, \ldots$$

- The property of being a subsequence is reflexive and transitive, though not symmetric:

- **Lemma 1.** Let $(a_n)_{n=0}^\infty$, $(b_n)_{n=0}^\infty$, and $(c_n)_{n=0}^\infty$ be sequences of real numbers. Then $(a_n)_{n=0}^\infty$ is a subsequence of $(a_n)_{n=0}^\infty$. Furthermore, if $(b_n)_{n=0}^\infty$ is a subsequence of $(a_n)_{n=0}^\infty$, and $(c_n)_{n=0}^\infty$ is a subsequence of $(b_n)_{n=0}^\infty$, then $(c_n)_{n=0}^\infty$ is a subsequence of $(a_n)_{n=0}^\infty$.

- **Proof.** See Week 6 homework. □

- Subsequences are also related to limits in the following way:

- **Proposition 2.** Let $(a_n)_{n=0}^\infty$ be a sequence of real numbers, and let L be a real number.

- (a) If the sequence $(a_n)_{n=0}^\infty$ converges to L, then every subsequence of $(a_n)_{n=0}^\infty$ also converges to L.

- (b) Conversely, if every subsequence of $(a_n)_{n=0}^\infty$ converges to L, then $(a_n)_{n=0}^\infty$ itself converges to L.

- **Proof.** See Week 6 homework. □

- Remember the concept of limit points from last week's notes? They correspond exactly to limits of subsequences (which explains why these points were called limit points in the first place):

- **Proposition 3.** Let $(a_n)_{n=0}^\infty$ be a sequence of real numbers, and let L be a real number.

- (a) Suppose L is a limit point of $(a_n)_{n=0}^\infty$. Then there exists a subsequence of $(a_n)_{n=0}^\infty$ which converges to L.

- (b) Conversely, suppose that there exists a subsequence of $(a_n)_{n=0}^\infty$ which converges to L. Then L is a limit point of $(a_n)_{n=0}^\infty$.

- **Proof.** See Week 6 homework. □

- Note the contrast between the notion of a limit, and that of a limit point, evidenced by Propositions 2 and 3. When a sequence has a limit L, then *all* subsequences also converge to L. But when a sequence only has L as a limit point, then only *some* subsequences converge to L.

- The above propositions, combined with some propositions from previous notes, have an important consequence: every bounded sequence has a convergent subsequence.

- **Bolzano-Weierstrass theorem.** Let $(a_n)_{n=0}^\infty$ be a bounded sequence (i.e. there exists a real number $M > 0$ such that $|a_n| \leq M$ for all $n \in \mathbf{N}$). Then there is at least one subsequence of $(a_n)_{n=0}^\infty$ which converges.

- **Proof.** Let L be the limit superior of the sequence $(a_n)_{n=0}^\infty$. Since we have $-M \leq a_n \leq M$ for all natural numbers n, it follows from the comparison principle (Lemma 28 from Week 3/4 notes) that $-M \leq L \leq M$. In particular, L is a real number (not $+\infty$ or $-\infty$). By Proposition 27(e) from Week 3/4 notes, L is thus a limit point of $(a_n)_{n=0}^\infty$. By Proposition 3(a), there thus exists a subsequence of $(a_n)_{n=0}^\infty$ which converges (in fact, it converges to L). □

- Note that we could as well have used the limit inferior instead of the limit superior in the above argument.

- The Bolzano-Weierstrass theorem says that if a sequence is bounded, then eventually it has no choice but to converge in some places; it has "no room" to spread out and stop itself from acquiring limit points. It is not true for unbounded sequences; for instance, the sequence $1, 2, 3, \ldots$ has no convergent subsequences whatsoever (why?). In the language of topology (which you will learn in Math 121), this means that the interval $\{x \in \mathbf{R} : -M \leq x \leq M\}$ is *compact*, whereas an unbounded set such as the real line $\mathbf{R}$ is not compact. The distinction between compact sets and non-compact sets will be very important in more advanced analysis courses - of similar importance to the distinction between finite sets and infinite sets. However, we will not cover this concept in depth here in this course.

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Functions on the real line

- In the last few weeks we have been focusing quite heavily on *sequences*. A sequence $(a_n)_{n=0}^{\infty}$ is something which assigns a real number a_n to each natural number n. In other words, it is a function from $\mathbf{N}$ to $\mathbf{R}$. We then did various things with these functions from $\mathbf{N}$ to $\mathbf{R}$, such as take their limit at infinity (if the function was convergent), or form suprema, infima, etc., or computed the sum of all the elements in the sequence (again, assuming the series was convergent).

- Now we will look at functions not on the natural numbers $\mathbf{N}$, which are "discrete", but instead look at functions on *continua* such as $\mathbf{R}$, or perhaps intervals such as $\{x \in \mathbf{R} : a \leq x \leq b\}$. (We will not define the notion of a discrete set or a continuum in this course, but roughly speaking a set is discrete if each element is separated from the rest of the set by some non-zero distance; at the other extreme, a set is a *continuum* if no element is separated from the rest of the set by a non-zero distance, and also every Cauchy sequence in the continuum converges in that continuum (this basically ensures that the continuum contains no "holes"; thus for instance the rationals are not considered a continuum).)

- You are familiar with many functions $f : \mathbf{R} \to \mathbf{R}$ from the real line to the real line. Some examples are: $f(x) := x^2 + 3x + 5$; $f(x) := 2^x/(x^2+1)$; $f(x) := \sin(x)\exp(x)$ (though of course we have not yet defined the functions sin and exp in this course). These are functions from $\mathbf{R}$ to $\mathbf{R}$ since to every real number x they assign a single real number $f(x)$. (Note that these functions are not necessarily injective or surjective). We can also consider more exotic functions, e.g.

$$f(x) := \begin{cases} 1 & \text{if } x \in \mathbf{Q} \\ 0 & \text{if } x \notin \mathbf{Q}. \end{cases}$$

  This function is not *algebraic* (i.e. it cannot be expressed in terms of x purely by using the standard algebraic operations of $+$, $-$, $\times$, $/$, $\surd$, etc.; we will not need this notion in this course), but it is still a function from $\mathbf{R}$ to $\mathbf{R}$, because it still assigns a real number $f(x)$ to each $x \in \mathbf{R}$.

- In the above cases, the *domain* of the function was all of the real line $\mathbf{R}$. However, we can work on smaller domains, such as the positive half-line $\mathbf{R}^+ := \{x \in \mathbf{R} : x > 0\}$, the negative half-line $\mathbf{R}^- := \{x \in \mathbf{R} : x < 0\}$, or on various (bounded) intervals. (Bounded) intervals come in four types: the *closed intervals*

$$[a,b] := \{x \in \mathbf{R} : a \leq x \leq b\},$$

  the two types of *half-open intervals*

$$[a,b) := \{x \in \mathbf{R} : a \leq x < b\}; \quad (a,b] := \{x \in \mathbf{R} : a < x \leq b\},$$

  and the *open intervals*

$$(a,b) := \{x \in \mathbf{R} : a < x < b\}.$$

  Here a and b are real numbers such that $a \leq b$. These are of course not the only subsets of $\mathbf{R}$ (consider, for instance, the natural numbers $\mathbf{N}$, or the rationals $\mathbf{Q}$), but they do appear very frequently in analysis.

- There are also the half-infinite intervals, which can be either closed half-intervals such as

$$[a,\infty) := \{x \in \mathbf{R} : x \geq a\} \text{ and } (-\infty,a] := \{x \in \mathbf{R} : x \leq a\}$$

and the open half-intervals such as

$$(a, \infty) := \{x \in \mathbf{R} : x > a\} \text{ and } (-\infty, a) := \{x \in \mathbf{R} : x < a\}$$

To complete the set, we sometimes refer to the entire real-line as the doubly-infinite interval $(-\infty, \infty)$.

- We can take any one of the previous functions $f : \mathbf{R} \to \mathbf{R}$ defined on all of $\mathbf{R}$, and *restrict* the domain to a smaller set $X \subseteq \mathbf{R}$, creating a new function, sometimes called $f|_X$, from X to $\mathbf{R}$. This is the same function as the original function f, but is only defined on a smaller domain. (Thus $f|_X(x) := f(x)$ when $x \in X$, and $f|_X(x)$ is undefined when $x \notin X$). For instance, we can restrict the function $f(x) := x^2$, which is initially defined from $\mathbf{R}$ to $\mathbf{R}$, to the interval $[1, 2]$, thus creating a new function $f|_{[1,2]} : [1, 2] \to \mathbf{R}$, which is defined as $f|_{[1,2]}(x) = x^2$ when $x \in [1, 2]$ but is undefined elsewhere.

- One could also restrict the range from $\mathbf{R}$ to some smaller subset Y of $\mathbf{R}$, provided of course that all the values of $f(x)$ lie inside $\mathbf{R}$. For instance, the function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) := x^2$ could also be thought of as a function from $\mathbf{R}$ to $[0, \infty)$, instead of a function from $\mathbf{R}$ to $\mathbf{R}$. Strictly speaking, these two functions are different functions, but the distinction between them is so minor that we shall often be careless about the range of a function in this week's discussion.

- Strictly speaking, there is a distinction to make between a *function* f, and its *value* $f(x)$ at a point x. f is a function; but $f(x)$ is a number (which depends on some free variable x). This distinction is rather subtle and we will not stress it too much, but there are times when one has to distinguish between the two. For instance, if $f : \mathbf{R} \to \mathbf{R}$ is the function $f(x) := x^2$, and $g := f|_{[1,2]}$ is the restriction of f to the interval $[1, 2]$, then f and g both perform the operation of squaring, i.e. $f(x) = x^2$ and $g(x) = x^2$, but the two functions f and g are not considered the same function, $f \neq g$, because they have different domains. Despite this distinction, we shall often be careless, and say things like "consider the function $x^2 + 2x + 3$" when really we should be saying "consider the function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) := x^2 + 2x + 3$". (This distinction makes more of a difference when we start doing things like differentiation. For instance, if $f : \mathbf{R} \to \mathbf{R}$ is the function $f(x) = x^2$, then of course $f(3) = 9$, but the derivative of f at 3 is 6, whereas the derivative of 9 is of course 0, so we cannot simply "differentiate both sides" of $f(3) = 9$ and conclude that $6 = 0$).

- We can represent functions from $\mathbf{R}$ to $\mathbf{R}$ (or from any subset of $\mathbf{R}$ to $\mathbf{R}$) by means of a graph in $\mathbf{R}^2$, as you all know; this is of course a very useful visual tool, though it is possible to work with functions without the aid of graphs.

- Given two functions $f : X \to Y$ and $g : Y \to Z$, we can form the composition $g \circ f : X \to Z$ defined by $g \circ f(x) := g(f(x))$ for all $x \in X$. Note that while g appears to the left of f in $g \circ f$, in fact f is applied to x *first*. This is an unfortunate blemish in modern mathematical notation (the problem is that the function f is always written before the variable x, when a more logical approach would have been to put the function after the variable (i.e. $(x)f$ instead of $f(x)$)), but it is too late to do anything about it now. Thus, for instance, if $f : \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$ are the functions $f(x) := x^2$ and $g(x) := 2x$, then $f \circ g(x) := (2x)^2 = 4x^2$, while $g \circ f(x) = 2(x^2) = 2x^2$.

- Given two functions $f : X \to \mathbf{R}$ and $g : X \to \mathbf{R}$, we can define their sum $f + g : X \to \mathbf{R}$ by the formula

$$(f + g)(x) := f(x) + g(x).$$

  Thus, for instance, if $f : \mathbf{R} \to \mathbf{R}$ is the function $f(x) := x^2$, and $g : \mathbf{R} \to \mathbf{R}$ is the function $g(x) := 2x$, then $f + g : \mathbf{R} \to \mathbf{R}$ is the function $(f + g)(x) := x^2 + 2x$. In a similar vein, we can define $f - g : X \to \mathbf{R}$ by

$$(f - g)(x) := f(x) - g(x),$$

  the product $fg : X \to \mathbf{R}$ by

$$(fg)(x) := f(x)g(x),$$

  and (provided that $g(x) \neq 0$ for all $x \in X$) the quotient $f/g : X \to \mathbf{R}$ by

$$(f/g)(x) := f(x)/g(x).$$

Finally, if $c \in \mathbf{R}$ is a real number, we can define $cf : X \to \mathbf{R}$ by

$$(cf)(x) := cf(x).$$

For instance, in the preceding example, $6f : \mathbf{R} \to \mathbf{R}$ is the function $(6f)(x) := 6x^2$.

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Adherent points of sets

- We now pause to define a technical concept - that of a *adherent point* of a set. This is similar to, but slightly different from, the notion of a limit point of a sequence, which we discussed earlier.

- **Definition** Let X be a subset of $\mathbf{R}$, let $\varepsilon > 0$, and let $x \in \mathbf{R}$. We say that x is *ε-adherent to X* iff there exists a $y \in X$ which is ε-close to x (i.e. $|x - y| \leq \varepsilon$).

- **Example.** The point 1.1 is 0.5-adherent to the open interval $(0, 1)$, but is not 0.1-adherent to this interval (why?). The point 1.1 is 0.5-adherent to the finite set $\{1, 2, 3\}$. The point 1 is 0.5-adherent to $\{1, 2, 3\}$ (why?).

- **Definition** Let X be a subset of $\mathbf{R}$, and let $x \in \mathbf{R}$. We say that x is an *adherent point* of X iff it is ε-adherent to X for every $\varepsilon > 0$.

- **Example** The number 1 is ε-adherent to the open interval $(0, 1)$ for every $\varepsilon > 0$ (why?), and is thus an adherent point of $(0, 1)$. The point 0.5 is similarly an adherent point of $(0, 1)$. However, the number 2 is not 0.5-adherent (for instance) to $(0, 1)$, and is thus not an adherent point to $(0, 1)$.

- **Example** Given any subset X of $\mathbf{R}$, every element of X is an adherent point of X (why?).

- **Example** Every real number x is an adherent point of $\mathbf{R}$ (why?). Every integer is of course an adherent point of $\mathbf{Z}$, but every non-integer is not an adherent point of $\mathbf{Z}$.

- **Example** Every real number x is an adherent point of $\mathbf{Q}$ (why?).

- A remark: there is a slightly different notion of a *limit point* or *cluster point* in the literature, which is a little bit different from adherent point, but we will not need these notions here. (You can look up limit points in page 32 of the textbook, though).

- **Definition** Let X be a subset of $\mathbf{R}$. The *closure* of X, sometimes denoted $\overline{X}$ is defined to be the set of all the adherent points of X.

- **Lemma 4.** Let $a < b$ be real numbers, and let I be any one of the four intervals (a, b), $(a, b]$, $[a, b)$, or $[a, b]$. Let x be a real number. Then the closure of I is $[a, b]$. Similarly, the closure of (a, ∞) or $[a, \infty)$ is $[a, \infty)$, while the closure of $(-\infty, a)$ or $(-\infty, a]$ is $(-\infty, a]$. Finally, the closure of $(-\infty, \infty)$ is $(-\infty, \infty)$.

- **Proof.** We will just show one of these facts, namely that the closure of (a, b) is $[a, b]$; the other results are proven similarly.

- First let us show that every element of $[a, b]$ is adherent to (a, b). Let $x \in [a, b]$. If $x \in (a, b)$ then it is definitely adherent to (a, b). If $x = b$ then x is also adherent to (a, b) (why?). Similarly when $x = a$. Thus every point in $[a, b]$ is adherent to (a, b).

- Now we show that every point x adherent to (a, b) lies in $[a, b]$. Suppose for contradiction that x does not lie in $[a, b]$, then either $x > b$ or $x < a$. If $x > b$ then x is not $(x - b)$-adherent to (a, b) (why?), and is hence not an adherent point to (a, b). Similarly if $x < a$. This contradiction shows that x is in fact in $[a, b]$ as claimed. □

- The following lemma shows that adherent points of a set X can be obtained as the limit of elements in X:

- **Lemma 5.** Let X be a subset of $\mathbf{R}$, and let $x \in \mathbf{R}$. Then x is an adherent point of X if and only if there exists a sequence $(a_n)_{n=0}^{\infty}$, consisting entirely of elements in X, which converges to x.

- **Proof.** See Week 6 homework. □

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Limiting values of functions

- In previous notes, we defined what it means for a sequence $(a_n)_{n=0}^{\infty}$ to converge to a limit L. We now define a similar notion for what it means for a function f defined on the real line, or on some subset of the real line, to converge to some value at a point.

- Just as we used the notions of ε-closeness and eventual ε-closeness to deal with limits of sequences, we shall need a notion of ε-closeness and local ε-closeness to deal with limits of functions.

- **Definition.** Let X be a subset of $\mathbf{R}$, let $f : X \to \mathbf{R}$ be a function, let L be a real number, and let $\varepsilon > 0$ be a real number. We say that the function f is *ε-close to L* iff $f(x)$ is ε-close to L for every $x \in X$.

- **Example.** When the function $f(x) := x^2$ is restricted to the interval $[1, 3]$, then it is 5-close to 4, since when $x \in [1, 3]$ then $1 \leq f(x) \leq 9$, and hence $|f(x) - 4| \leq 5$. When instead it is restricted to the smaller interval $[1.9, 2.1]$, then it is 0.41-close to 4, since if $x \in [1.9, 2.1]$, then $3.61 \leq f(x) \leq 4.41$, and hence $|f(x) - 4| \leq 0.41$.

- **Definition.** Let X be a subset of $\mathbf{R}$, let $f : X \to \mathbf{R}$ be a function, let L be a real number, x_0 be an adherent point of X, and $\varepsilon > 0$ be a real number. We say that f is *ε-close to L near x_0* iff there exists a $\delta > 0$ such that f becomes ε-close to L when restricted to the set $\{x \in X : |x - x_0| < \delta\}$.

- **Example.** Let $f : [1, 3] \to \mathbf{R}$ be the function $f(x) := x^2$, restricted to the interval $[1, 3]$. This function is not 0.1-close to 4, since for instance $f(1)$ is not 0.1-close to 4. However, f is 0.1-close to 4 near 2, since when restricted the set $\{x \in [1, 3] : |x - 2| < 0.01\}$, the function f is indeed 0.1-close to 4 (since when $|x - 2| < 0.01$, we have $1.99 < x < 2.01$, and hence $3.9601 < f(x) < 4.0401$, and in particular $f(x)$ is 0.1-close to 4).

- **Example.** Continuing with the same function f used in the previous example, we observe that f is not 0.1-close to 9, since for instance $f(1)$ is not 0.1-close to 9. However, f is 0.1-close to 9 near 3, since when restricted to the set $\{x \in [1, 3] : |x - 3| < 0.01\}$ - which is the same as the half-open interval $(2.99, 3]$ (why?), the function f becomes 0.1-close to 9 (since if $2.99 < x \leq 3$, then $8.9401 < f(x) \leq 9$, and hence $f(x)$ is 0.1-close to 9.).

- **Definition.** Let X be a subset of $\mathbf{R}$, let $f : X \to \mathbf{R}$ be a function, let E be a subset of X, x_0 be an adherent point of E, and let L be a real number. We say that f *converges to* L *at* x_0 *in* E, and write $\lim_{x \to x_0; x \in E} f(x) = L$, iff f is ε-close to L near x_0 for every $\varepsilon > 0$. If f does not converge to any number L at x_0, we say that f *diverges* at x_0, and leave $\lim_{x \to x_0; x \in E} f(x)$ undefined.

- In other words, we have $\lim_{x \to x_0; x \in E} f(x) = L$ iff for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - L| \leq \varepsilon$ for all $x \in E$ such that $|x - x_0| < \delta$. (Why is this definition equivalent to the one given above?).

- In many cases we will omit the set E from the above notation (i.e. we will just say that f converges to L at x_0, or that $\lim_{x \to x_0} f(x) = L$), although this is slightly dangerous. More on this later.

- **Example.** Let $f : [1, 3] \to \mathbf{R}$ be the function $f(x) := x^2$. We have seen before that f is 0.1-close to 4 near 2. A similar argument shows that f is 0.01-close to 4 near 2 (one just has to pick a smaller value of δ).

- This definition is rather unwieldy. However, we can rewrite this definition in terms of a more familiar one, involving limits of sequences.

- **Proposition 6.** Let X be a subset of $\mathbf{R}$, let $f : X \to \mathbf{R}$ be a function, let E be a subset of X, let x_0 be an adherent point of E, and let L be a real number. Then the following two statements are equivalent (i.e. if one is true then the other is true, and conversely):

- (a) f converges to L at x_0 in E.

- (b) For every sequence $(a_n)_{n=0}^{\infty}$ which consists entirely of elements of E, which converges to x_0, the sequence $(f(a_n))_{n=0}^{\infty}$ converges to $f(x_0)$.

- **Proof.** See Week 6 homework. □

- One consequence of Proposition 6 can be written somewhat informally as follows: if $\lim_{x \to x_0; x \in E} f(x) = L$, and $\lim_{n \to \infty} a_n = x_0$, then $\lim_{n \to \infty} f(a_n) = L$. (Why does this follow from Proposition 6? I said that this statement was somewhat informal, because I didn't specify all the assumptions on f, x_0, L, E, etc.).

- Because of the above proposition, we will sometimes say "$f(x) \to L$ as $x \to x_0$ in E" or "f has limit L at x_0 in E" instead of "f converges to L at x_0", or "$\lim_{x \to x_0} f(x) = L$".

- Note that we only consider limits of a function f at x_0 in the case when x_0 is an adherent point of E. When x_0 is not an adherent point then it is not worth it to define the concept of a limit (can you see why there will be problems?).

- Note that the variable x used to denote limit is a dummy variable; we could replace it by any other variable and obtain exactly the same limit. For instance, if $\lim_{x \to x_0; x \in E} f(x) = L$, then $\lim_{y \to x_0; y \in E} f(y) = L$, and conversely (why?).

- Proposition 6 has some immediate corollaries. For instance, we now know that a function can have at most one limit at each point.

- **Corollary 7.** Let X be a subset of $\mathbf{R}$, let E be a subset of X, let x_0 be an adherent point of E, and let $f : X \to \mathbf{R}$ be a function. Then f can have at most one limit at x_0 in E.

- **Proof.** Suppose for contradiction that there are two distinct limits L and L' such that f has limit L at x_0 in E, and such that f also has limit L' at x_0 in E. Since x_0 is an adherent point of E, we know by Lemma 5 that there is a sequence $(a_n)_{n=0}^{\infty}$ consisting of elements in E which converges to x_0. Since f has limit L at x_0 in E, we thus see by Proposition 6, that $(f(a_n))_{n=0}^{\infty}$ converges to L. But since f also has limit L' at x_0 in E, we see that $(f(a_n))_{n=0}^{\infty}$ also converges to L'. But this contradicts the uniqueness of limits of sequences (Proposition 16 of Week 3/4 notes). □

- Using the limit laws for sequences, one can now deduce the limit laws for functions:

- **Proposition 8. (Limit laws for functions)** Let X be a subset of $\mathbf{R}$, let E be a subset of X, let x_0 be an adherent point of E, and let $f : X \to \mathbf{R}$ and $g : X \to \mathbf{R}$ be functions. Suppose that f has limit L at x_0 in E, and g has limit M at x_0 in E. Then $f + g$ has limit $L + M$ at x_0 in E, $f - g$ has limit $L - M$ at x_0 in E, and fg has limit LM at

x_0 in E. If c is a real number, then cf has limit cL at x_0 in E. Finally, if g is non-zero on E (i.e. $g(x) \neq 0$ for all $x \in E$) and M is non-zero, then f/g has limit L/M at x_0 in E.

- **Proof.** We just prove the first claim (that $f+g$ has limit $L+M$); the others are very similar and are left to the reader. Since x_0 is an adherent point of E, we know by Lemma 5 that there is a sequence $(a_n)_{n=0}^{\infty}$ consisting of elements in E which converge to x_0. Since f has limit L at x_0 in E, we thus see by Proposition 6, that $(f(a_n))_{n=0}^{\infty}$ converges to L. Similarly $(g(a_n))_{n=0}^{\infty}$ converges to M. By the limit laws for sequences (Theorem 21 from Week 3/4 notes) we conclude that $((f+g)(a_n))_{n=0}^{\infty}$ converges to $L+M$. By Proposition 6 again, this implies that $f+g$ has limit $L+M$ at x_0 in E as desired (since $(a_n)_{n=0}^{\infty}$ was an arbitrary sequence in E converging to x_0). The other claims are proven similarly. □

- One can phrase Proposition 8 more informally as saying that

$$\lim_{x \to x_0; x \in E} (f+g)(x) = \lim_{x \to x_0; x \in E} f(x) + \lim_{x \to x_0; x \in E} g(x)$$

$$\lim_{x \to x_0; x \in E} (f-g)(x) = \lim_{x \to x_0; x \in E} f(x) - \lim_{x \to x_0; x \in E} g(x)$$

$$\lim_{x \to x_0; x \in E} (fg)(x) = \lim_{x \to x_0; x \in E} f(x) \lim_{x \to x_0; x \in E} g(x)$$

$$\lim_{x \to x_0; x \in E} (f/g)(x) = \frac{\lim_{x \to x_0; x \in E} f(x)}{\lim_{x \to x_0; x \in E} g(x)}$$

  but bear in mind that these identities are only true when the right-hand side makes sense, and furthermore for the third identity we need g to be non-zero, and also $\lim_{x \to x_0; x \in E} g(x)$ to be non-zero.

- Using these limit laws we can already deduce several limits. First of all, it is easy to check the basic limits

$$\lim_{x \to x_0; x \in \mathbf{R}} c = c$$

  and

$$\lim_{x \to x_0; x \in \mathbf{R}} x = x_0$$

for any real numbers x_0 and c (why? use Proposition 6). By the limit laws we can thus conclude that

$$\lim_{x \to x_0; x \in \mathbf{R}} x^2 = x_0^2$$

$$\lim_{x \to x_0; x \in \mathbf{R}} cx = cx_0$$

$$\lim_{x \to x_0; x \in \mathbf{R}} x^2 + cx + d = x_0^2 + cx_0 + d$$

etc., where c, d are arbitrary real numbers.

- If f converges to L at x_0 in X, and Y is any subset of X such that x_0 is still an adherent point of Y, then f will also converge to L at x_0 in Y (why?). Thus convergence on a large set implies convergence on a smaller set. The converse, however, is not true. Consider the signum function $\mathrm{sgn} : \mathbf{R} \to \mathbf{R}$, defined by

$$\mathrm{sgn}(x) := \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

Then $\lim_{x \to 0 : x \in (0,\infty)} \mathrm{sgn}(x) = 1$ (why?), but $\lim_{x \to 0 : x \in (-\infty,0)} = -1$ (why?), while $\lim_{x \to 0 : x \in \mathbf{R}}$ is undefined (why?). Thus it is sometimes dangerous to drop the set X from the notation of limit. However, in many cases it is safe to do so; for instance, since we know that $\lim_{x \to x_0; x \in \mathbf{R}} x^2 = x_0^2$, we know in fact that $\lim_{x \in x_0; x \in X} x^2 = x_0^2$ for any set X with x_0 as an adherent point (why?). Thus it is safe to write $\lim_{x \to x_0} x^2 = x_0^2$.

- **Example.** Let $f(x)$ be the function

$$f(x) := \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0. \end{cases}$$

Then $\lim_{x \to 0 : x \in \mathbf{R} - \{0\}} f(x) = 0$ (why?), but $\lim_{x \to 0 : x \in \mathbf{R}} f(x)$ is undefined (why). (When this happens, we say that f has a "removable singularity" or "removable discontinuity" at 0. Because of such singularities, it is sometimes the convention when writing $\lim_{x \to x_0} f(x)$ to automatically exclude x_0 from the set; for instance, in the textbook, $\lim_{x \to x_0} f(x)$ is used as shorthand for $\lim_{x \to x_0 : x \in X - \{x_0\}} f(x)$).

- On the other hand, the limit at x_0 should only depend on the values of the function near x_0; the values away from x_0 are not relevant. The following proposition reflects this intuition:

- **Proposition 9.** Let X be a subset of $\mathbf{R}$, let E be a subset of X, let x_0 be an adherent point of E, let $f : X \to \mathbf{R}$ be a function, and let L be a real number. Let $\delta > 0$. Then we have

$$\lim_{x \to x_0 : x \in E} f(x) = L$$

if and only if

$$\lim_{x \to x_0 : x \in E \cap (x_0 - \delta, x_0 + \delta)} f(x) = L.$$

- **Proof.** See Week 6 homework. □

- Informally, the above proposition asserts that

$$\lim_{x \to x_0 ; x \in E} f(x) = \lim_{x \to x_0 ; x \in E \cap (x_0 - \delta, x_0 + \delta)} f(x).$$

Thus the limit of a function at x_0, if it exists, only depends on the values of f near x_0; the values far away do not actually influence the limit.

- We now give a few more examples of limits.

- **Example** Consider the functions $f : \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$ defined by $f(x) := x + 2$ and $g(x) := x + 1$. Then $\lim_{x \to 2; x \in \mathbf{R}} f(x) = 4$ and $\lim_{x \to 2; x \in \mathbf{R}} g(x) = 3$. We would like to use the limit laws to conclude that $\lim_{x \to 2; x \in \mathbf{R}} f(x)/g(x) = 4/3$, or in other words that $\lim_{x \to 2; x \in \mathbf{R}} \frac{x+2}{x+1} = \frac{4}{3}$. Strictly speaking, we cannot use Proposition 8 to ensure this, because $x + 1$ is zero at $x = -1$, and so $f(x)/g(x)$ is not defined. However, this is easily solved, by restricting the domain of f and g from $\mathbf{R}$ to a smaller domain, such as $\mathbf{R} - \{1\}$. Then Proposition 8 does apply, and we have $\lim_{x \to 2; x \in \mathbf{R} - \{1\}} \frac{x+2}{x+1} = \frac{4}{3}$.

- **Example** Consider the function $f : \mathbf{R} - \{1\} \to \mathbf{R}$ defined by $f(x) := (x^2 - 1)/(x - 1)$. This function is well-defined for every real number except 1, so $f(1)$ is undefined. However, 1 is still an adherent point of

$\mathbf{R} - \{1\}$ (why?), and the limit $\lim_{x \to 1; x \in \mathbf{R} - \{1\}} f(x)$ is still defined. This is because on the domain $\mathbf{R} - \{1\}$ we have the identity $(x^2 - 1)/(x - 1) = (x+1)(x-1)/(x-1) = x + 1$, and $\lim_{x \to 1; x \in \mathbf{R} - \{1\}} x + 1 = 2$.

- **Example** Let $f : \mathbf{R} \to \mathbf{R}$ be the function

$$f(x) := \begin{cases} 1 & \text{if } x \in \mathbf{Q} \\ 0 & \text{if } x \notin \mathbf{Q}. \end{cases}$$

We will show that $f(x)$ has no limit at 0 in $\mathbf{R}$. Suppose for contradiction that $f(x)$ had some limit L at 0 in $\mathbf{R}$. Then we would have $\lim_{n \to \infty} f(a_n) = L$ whenever $(a_n)_{n=0}^\infty$ is a sequence of non-zero numbers converging to 0. Since $(1/n)_{n=0}^\infty$ is such a sequence, we would have

$$L = \lim_{n \to \infty} f(1/n) = \lim_{n \to \infty} 1 = 1.$$

On the other hand, since $(\sqrt{2}/n)_{n=0}^\infty$ is another sequence of non-zero numbers converging to 0 - but now these numbers are irrational instead of rational - we have

$$L = \lim_{n \to \infty} f(\sqrt{2}/n) = \lim_{n \to \infty} 0 = 0.$$

Since $1 \neq 0$, we have a contradiction. Thus this function does not have a limit at 0.

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Continuous functions

- We now introduce one of the most fundamental notions in the theory of functions - that of *continuity*.

- **Definition.** Let X be a subset of $\mathbf{R}$, and let $f : X \to \mathbf{R}$ be a function. Let x_0 be an element of X. We say that f is *continuous at* x_0 iff we have

$$\lim_{x \to x_0; x \in X} f(x) = f(x_0);$$

in other words, the limit of $f(x)$ as x converges to x_0 in X exists and is equal to $f(x_0)$. We say that f is *continuous on* X (or simply *continuous*) iff f is continuous at x_0 for every $x_0 \in X$. We say that f is *discontinuous at* x_0 iff it is not continuous at x_0.

- **Example.** Let c be a real number, and let $f : \mathbf{R} \to \mathbf{R}$ be the constant function $f(x) := c$. Then for every real number $x_0 \in \mathbf{R}$, we have $\lim_{x_0 \to x; x \in \mathbf{R}} f(x) = \lim_{x_0 \in x; x \in \mathbf{R}} c = c = f(x_0)$, thus f is continuous at every point $x_0 \in \mathbf{R}$, and thus f is continuous on $\mathbf{R}$.

- **Example.** Let $f : \mathbf{R} \to \mathbf{R}$ be the identity function $f(x) := x$. Then for every real number $x_0 \in \mathbf{R}$, we have $\lim_{x \to x_0; x \in \mathbf{R}} f(x) = \lim_{x_0 \in x; x \in \mathbf{R}} x = x_0 = f(x_0)$, thus f is continuous at every point $x_0 \in \mathbf{R}$, and thus f is continuous on $\mathbf{R}$.

- **Example.** Let $\mathrm{sgn} : \mathbf{R} \to \mathbf{R}$ be the signum function defined earlier. Then $\mathrm{sgn}(x)$ is continuous at every non-zero value of x; for instance, at 1, we have (using Proposition 9)

$$\lim_{x \to 1; x \in \mathbf{R}} \mathrm{sgn}(x) = \lim_{x \to 1; x \in (0.9, 1.1)} \mathrm{sgn}(x) = \lim_{x \to 1; x \in (0.9, 1.1)} 1 = 1 = \mathrm{sgn}(1).$$

  On the other hand, sgn is not continuous at 0, since $\lim_{x \to 0; x \in \mathbf{R}} \mathrm{sgn}(x)$ does not exist.

- **Example** Let $f : \mathbf{R} \to \mathbf{R}$ be the function

$$f(x) := \begin{cases} 1 & \text{if } x \in \mathbf{Q} \\ 0 & \text{if } x \notin \mathbf{Q}. \end{cases}$$

  Then by the discussion in the previous section, f is not continuous at 0. In fact, it turns out that f is not continuous at any real number x_0 (can you see why?).

- **Example** Let $f : \mathbf{R} \to \mathbf{R}$ be the function

$$f(x) := \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases}$$

  Then f is continuous at every non-zero real number (why?), but is not continuous at 0. However, if we restrict f to the right-hand line $[0, \infty)$, then the resulting function $f|_{[0,\infty)}$ now becomes continuous everywhere in its domain, including 0. Thus restricting the domain of a function can make a discontinuous function continuous again.

- There are several ways to phrase the statement that "f is continuous at x_0":

- **Proposition 10.** Let X be a subset of $\mathbf{R}$, let $f : X \to \mathbf{R}$ be a function, and let x_0 be an element of X. Then the following three statements are equivalent:

- (a) f is continuous at x_0.

- (b) For every sequence $(a_n)_{n=0}^{\infty}$ consisting of elements of X with $\lim_{n\to\infty} a_n = x_0$, we have $\lim_{n\to\infty} f(a_n) = f(x_0)$.

- (c) For every $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - f(x_0)| < \varepsilon$ for all $x \in X$ with $|x - x_0| < \delta$.

- **Proof.** See Week 6 homework. $\square$

- A particularly useful consequence of Proposition 10 is the following: if f is continuous at x_0, and $a_n \to x_0$ as $n \to \infty$, then $f(a_n) \to f(x_0)$ as $n \to \infty$ (provided that all the elements of the sequence $(a_n)_{n=0}^{\infty}$ lie in the domain of f, of course). Thus continuous functions are very useful in computing limits.

- The limit laws in Proposition 8, combined with the definition of continuity, immediately implies

- **Proposition 11.** Let X be a subset of $\mathbf{R}$, and let $f : X \to \mathbf{R}$ and $g : X \to \mathbf{R}$ be functions. Let $x_0 \in X$. Then if f and g are both continuous at x_0, then the functions $f + g$ and fg are also continuous at x_0. If g is non-zero on X, then f/g is also continuous at x_0.

- In particular, the sum, difference, and product of continuous functions are continuous; and the quotient of two continuous functions is continuous as long as the denominator does not become zero.

- One can use Proposition 11 to show that a lot of functions are continuous. For instance, just by starting from the fact that constant functions are continuous, and the identity function $f(x) = x$ is continuous, one can show that the function $g(x) := (x^3 + 4x^2 + x + 5)/(x^2 - 4)$, for instance, is continuous at every point of $\mathbf{R}$ except the two points $x = +2$, $x = -2$ where the denominator vanishes.

- Some other examples of continuous functions:
- **Proposition 12.** Let $a > 0$ be a positive real number. Then the function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) := a^x$ is continuous.
- **Proof.** See Week 6 homework. $\square$
- **Proposition 13.** Let p be a real number. Then the function $f : (0, \infty) \to \mathbf{R}$ defined by $f(x) := x^p$ is continuous.
- **Proof.** See Week 6 homework. $\square$
- **Proposition 14.** The function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) := |x|$ is continuous.
- **Proof.** We have to show that f is continuous at every real number x_0. So, let x_0 be an arbitrary real number. There are three cases, $x_0 > 0$, $x_0 = 0$, and $x_0 < 0$. First suppose that $x_0 > 0$; we have to show that $\lim_{x \to x_0; x \in \mathbf{R}} |x| = |x_0|$. By Proposition 9 (with $\delta := x_0/2$), we have

  $$\lim_{x \to x_0; x \in \mathbf{R}} |x| = \lim_{x \to x_0; x \in (x_0/2, 3x_0/2)} |x| = \lim_{x \to x_0; x \in (x_0/2, 3x_0/2)} x = x_0 = |x_0|$$

  as desired. Thus f is continuous at x_0 when x_0 is positive. A similar argument gives continuity when x is negative; so it suffices to show that f is continuous at 0. By Proposition 10, it suffices to show that whenever $(a_n)_{n=0}^\infty$ is a sequence of real numbers converging to 0, then $(|a_n|)_{n=0}^\infty$ also converges to $|0| = 0$. But this follows from Q10 of Week 4 homework. $\square$
- The class of continuous functions are not only closed under addition, subtraction, multiplication, and division, but are also closed under composition:
- **Proposition 15.** Let X and Y be subsets of $\mathbf{R}$, and let $f : X \to Y$ and $g : Y \to \mathbf{R}$ be functions. Let x_0 be a point in X. If f is continuous at x_0, and g is continuous at $f(x_0)$, then the composition $g \circ f : X \to \mathbf{R}$ is continuous at x_0.
- **Proof.** See Week 6 homework. $\square$.

- **Example** Since the function $f(x) := 3x + 1$ is continuous on all of $\mathbf{R}$, and the function $g(x) := 5^x$ is continuous on all of $\mathbf{R}$, the function $g \circ f(x) = 5^{3x+1}$ is continuous on all of $\mathbf{R}$. By several applications of the above propositions, one can show that far more complicated functions, e.g. $h(x) := |x^2 - 8x + 7|^{\sqrt{2}}/(x^2 + 1)$, are also continuous (why is this function continuous?). There are still a few functions though that are not yet easy to test for continuity, such as $k(x) := x^x$; this function can be dealt with more easily once we have the machinery of logarithms, which we will see later in this course.

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Left and right limits

- We now introduce the notion of left and right limits, which can be thought of as two seperate "halves" of the complete limit $\lim_{x \to x_0; x \in X} f(x)$.

- **Definition.** Let X be a subset of $\mathbf{R}$, $f : X \to \mathbf{R}$ be a function, and let x_0 be a real number. If x_0 is an adherent point of $X \cap (x_0, \infty)$, then we define the *right limit* $f(x_0+)$ of f at x_0 by the formula

$$f(x_0+) := \lim_{x \to x_0; x \in X \cap (x_0, \infty)} f(x);$$

  if this limit does not exist, or x_0 is not an adherent point of $X \cap (x_0, \infty)$, we leave $f(x_0+)$ undefined. Similarly, if x_0 is an adherent point of $X \cap (-\infty, x_0)$, then we define the *left limit* $f(x_0-)$ of f at x_0 by the formula

$$f(x_0-) := \lim_{x \to x_0; x \in X \cap (-\infty, x_0)} f(x);$$

  if this limit does not exist, or x_0 is not an adherent point of $X \cap (-\infty, x_0)$, we leave $f(x_0-)$ undefined.

- Sometimes we write $\lim_{x \to x_0+} f(x)$ as shorthand for $\lim_{x \to x_0; x \in X \cap (x_0, \infty)} f(x)$, and similarly write $\lim_{x \to x_0-} f(x)$ as shorthand for $\lim_{x \to x_0; x \in X \cap (-\infty, x_0)} f(x)$.

- **Example** Consider the signum function $\mathrm{sgn} : \mathbf{R} \to \mathbf{R}$ defined earlier. We have

$$\mathrm{sgn}(0+) = \lim_{x \to x_0; x \in \mathbf{R} \cap (0, \infty)} \mathrm{sgn}(x) = \lim_{x \to x_0; x \in \mathbf{R} \cap (0, \infty)} 1 = 1$$

while

$$\text{sgn}(0-) = \lim_{x \to x_0; x \in \mathbf{R} \cap (-\infty, 0)} \text{sgn}(x) = \lim_{x \to x_0; x \in \mathbf{R} \cap (-\infty, 0)} -1 = -1,$$

while $\text{sgn}(0) = 0$ by definition.

- Note that f does not necessarily have to be defined at x_0 in order for $f(x_0+)$ or $f(x_0-)$ to be defined. For instance, if $f : \mathbf{R} - \{0\} \to \mathbf{R}$ is the function $f(x) := x/|x|$, then $f(0+) = 1$ and $f(0-) = -1$ (why?), even though $f(0)$ is undefined.

- From Proposition 10 we see that if $f(x_0+)$ exists, and $(a_n)_{n=0}^{\infty}$ is a sequence in X converging to x_0 from the right (i.e. $a_n > x_0$ for all $n \in \mathbf{N}$), then $\lim_{n\to\infty} f(a_n) = f(x_0+)$. Similarly, if $(b_n)_{n=0}^{\infty}$ is a sequence converging to x_0 from the left (i.e. $a_n < x_0$ for all $n \in \mathbf{N}$) then $\lim_{n\to\infty} f(a_n) = f(x_0-)$.

- Let x_0 be an adherent point of both $X \cap (x_0, \infty)$ and $X \cap (-\infty, x_0)$. If f is continuous at x_0, it is clear from Proposition 10 that $f(x_0+)$ and $f(x_0-)$ both exist and are equal to $f(x_0)$ (can you see why?). A converse is also true (compare this with Proposition 27(f) of Week 3/4 notes):

- **Proposition 16.** Let X be a subset of $\mathbf{R}$ containing a real number x_0, and suppose that x_0 is an adherent point of both $X \cap (x_0, \infty)$ and $X \cap (-\infty, x_0)$. Let $f : X \to \mathbf{R}$ be a function. If $f(x_0+)$ and $f(x_0-)$ both exist and are both equal to $f(x_0)$, then f is continuous at x_0.

- **Proof.** Let us write $L := f(x_0)$. Then by hypothesis we have

$$\lim_{x \to x_0; x \in X \cap (x_0, \infty)} f(x) = L$$

and

$$\lim_{x \to x_0; x \in X \cap (-\infty, x_0)} f(x) = L.$$

Let $\varepsilon > 0$. From the first limit and Proposition 10, we know that there exists a $\delta_+ > 0$ such that $|f(x) - L| < \varepsilon$ for all $x \in X \cap (x_0, \infty)$ for which $|x - x_0| < \delta_+$. From the second limit we similarly know that there exists a $\delta_- > 0$ such that $|f(x) - L| < \varepsilon$ for all $x \in X \cap (-\infty, x_0)$ for which $|x - x_0| < \delta_-$. Now let $\delta := \min(\delta_-, \delta_+)$; then $\delta > 0$ (why?), and suppose that $x \in X$ is such that $|x - x_0| < \delta$. Then there are three cases: $x > x_0$, $x = x_0$, and $x < x_0$, but in all three cases we know that $|f(x) - L| < \varepsilon$ (why? the reason is different in each of the three cases). By Proposition 10 we thus have that f is continuous at x_0, as desired. $\square$

- As we saw with the signum function, it is possible for the left and right limits $f(x_0-)$, $f(x_0+)$ of a function f at a point x_0 to both exist, but not be equal each other; when this happens, we say that f has a *jump discontinuity* at x_0. Thus, for instance, the signum function has a jump discontinuity at zero. Also, it is possible for the left and right limits $f(x_0-)$, $f(x_0+)$ to exist and be equal each other, but not be equal to $f(x_0)$; when this happens we say that f has a *removable discontinuity* (or *removable singularity*) at x_0. For instance, if we take $f : \mathbf{R} \to \mathbf{R}$ be the function
$$f(x) := \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0, \end{cases}$$
then $f(0+)$ and $f(0-)$ both exist and equal 0 (why?), but $f(0)$ equals 1; thus f has a removable discontinuity at 0.

- (Optional remark) Jump discontinuities and removable discontinuities are not the only way a function can be discontinuous. Another way is for a function to go off to infinity at the discontinuity: for instance, the function $f : \mathbf{R} - \{0\} \to \mathbf{R}$ defined by $f(x) := 1/x$ has a discontinuity at 0 which is neither a jump discontinuity or a removable singularity; informally, $f(x)$ converges to $+\infty$ when x approaches 0 on the right, and converges to $-\infty$ when x approaches 0 on the left. (We have not defined precisely what it means to converge to $+\infty$ or $-\infty$; but see page 55 of the textbook). These types of singularities are sometimes known as *asymptotic discontinuities*. There are also *oscillatory discontinuities*, where the function remains bounded but still does not have a limit near x_0; for instance, the function $f : \mathbf{R} \to \mathbf{R}$ defined by
$$f(x) := \begin{cases} 1 & \text{if } x \in \mathbf{Q} \\ 0 & \text{if } x \notin \mathbf{Q} \end{cases}$$

has an oscillatory discontinuity at 0 (and in fact at any other real number also), since the function does not have either left or right limits at 0, even though the function remains bounded.

- The study of discontinuities (also called *singularities*) continues further, but is beyond the scope of this course. The subject is also taken up (but with a different perspective) in Math 132, Complex Analysis.