Math 131AH - Week 5
Textbook pages: 58-69. (Optional reading: 70-75).
Topics covered:

- Some standard limits
- Series; conditional versus absolute convergence
- Some series convergence tests
- The root and ratio tests

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More on exponentiation

- We will continue our study of sequences and series momentarily, but first we need to expand our notion of exponentiation some more. Currently we have only defined exponentiation x^n when x is a real number and n is an integer (and when n is negative we have to ensure x is non-zero). We would like to expand this definition to define x^y for all real numbers x and y, though as we will see this is only really feasible for *positive* x. (To raise negative numbers to non-integer powers requires the complex numbers, which are beyond the scope of this course; but see Math 132).
- As in last week's notes, we shall now use all the normal rules of algebra without further comment.
- We begin with the notion of an n^{th} root, which we can define using our notion of supremum.
- **Definition.** Let $x > 0$ be a positive real, and let $n \geq 1$ be a positive integer. We define $x^{1/n}$, also known as the n^{th} *root of* x, by the formula

$$x^{1/n} := \sup\{y \in \mathbf{R} : y \geq 0 \text{ and } y^n \leq x\}.$$

We first show that this supremum is not infinite:

- **Lemma 1.** Let $x > 0$ be a positive real, and let $n \geq 1$ be a positive integer. Then the set $E := \{y \in \mathbf{R} : y \geq 0 \text{ and } y^n \leq x\}$ is non-empty and is also bounded above. In particular, $x^{1/n}$ is a real number.

- **Proof.** The set E contains 0 (why?), so it is certainly not empty. Now we show it has an upper bound. We divide into two cases: $x \leq 1$ and $x > 1$. First suppose that we are in the case where $x \leq 1$. Then we claim that the set E is bounded above by 1. To see this, suppose for contradiction that there was an element $y \in E$ for which $y > 1$. But then $y^n > 1$ (why?), and hence $y^n > x$, a contradiction. Thus E has an upper bound. Now suppose that we are in the case where $x > 1$. Then we claim that the set E is bounded above by x. To see this, suppose for contradiction that there was an element $y \in E$ for which $y > x$. Since $x > 1$, we thus have $y > 1$. Since $y > x$ and $y > 1$, we have $y^n > x$ (why?), a contradiction. Thus in both cases E has an upper bound, and so $x^{1/n}$ is finite. □

- We list some basic properties of n^{th} root below.

- **Lemma 2.** Let $x, y > 0$ be positive reals, and let $n, m \geq 1$ be positive integers.

- (a) If $y = x^{1/n}$, then $y^n = x$.

- (b) Conversely, if $y^n = x$, then $y = x^{1/n}$.

- (c) $x^{1/n}$ is a positive real number.

- (d) We have $x > y$ if and only if $x^{1/n} > y^{1/n}$.

- (e) If $x > 1$, then $x^{1/k}$ is a decreasing function of k. If $x < 1$, then $x^{1/k}$ is an increasing function of k. If $x = 1$, then $x^{1/k} = 1$ for all k.

- (f) We have $(xy)^{1/n} = x^{1/n}y^{1/n}$.

- (g) We have $(x^{1/n})^{1/m} = x^{1/nm}$.

- **Proof.** See Week 5 homework. □

- The observant reader may note that this definition of $x^{1/n}$ might possibly be inconsistent with our previous notion of x^n when $n = 1$, but it is easy to check that $x^{1/1} = x = x^1$ (why?), so there is in fact no inconsistency.

- One consequence of Lemma 2(b) is the following cancellation law: if y and z are positive and $y^n = z^n$, then $y = z$. (Why does this follow from Lemma 2(b)?). Note that this only works when y and z are positive; for instance, $(-3)^2 = 3^2$, but we cannot conclude from this that $-3 = 3$.

- Now we define how to raise a positive number x to a *rational* exponent q.

- **Definition.** Let $x > 0$ be a positive real number, and let q be a rational number. To define x^q, we write $q = a/b$ for some integer a and positive integer b, and define
$$x^q := (x^{1/b})^a.$$

- Note that every rational q, whether positive, negative, or zero, can be written in the form a/b where a is an integer and b is positive (why?). However, the rational number q can be expressed in the form a/b in more than one way, for instance $1/2$ can also be expressed as $2/4$ or $3/6$. So to ensure that this definition is well-defined, we need to check that different expressions a/b give the same formula for x^q:

- **Lemma 3.** Let a, a' be integers and b, b' be positive integers such that $a/b = a'/b'$, and let x be a positive real number. Then we have $(x^{1/b'})^{a'} = (x^{1/b})^a$.

- **Proof.** There are three cases: $a = 0$, $a > 0$, $a < 0$. If $a = 0$, then we must have $a' = 0$ (why?) and so both $(x^{1/b'})^{a'}$ and $(x^{1/b})^a$ are equal to 1, so we are done.

- Now suppose that $a > 0$. Then $a' > 0$ (why?), and $ab' = ba'$. Write $y := x^{1/(ab')} = x^{1/(ba')}$. By Lemma 2(g) we have $y = (x^{1/b'})^{1/a}$ and $y = (x^{1/b})^{1/a'}$; by Lemma 2(a) we thus have $y^a = x^{1/b'}$ and $y^{a'} = x^{1/b}$. Thus we have
$$(x^{1/b'})^{a'} = (y^a)^{a'} = y^{aa'} = (y^{a'})^a = (x^{1/b})^a$$
as desired.

- Finally, suppose that $a < 0$. Then we have $(-a)/b = (-a')/b$. But $-a$ is positive, so the previous case applies and we have $(x^{1/b'})^{-a'} = (x^{1/b})^{-a}$. Taking the reciprocal of both sides we obtain the result. $\square$

- Thus x^q is well-defined for every rational q. Note that this new definition is consistent with our old definition for $x^{1/n}$ (why?) and is also consistent with our old definition for x^n (why?).

- Some basic facts about rational exponentiation:

- **Lemma 4.** Let $x, y > 0$ be positive reals, and let q, r be rationals.

- (a) x^q is a positive real.

- (b) We have $x^{q+r} = x^q x^r$ and $(x^q)^r = x^{qr}$.

- (c) We have $x^{-q} = 1/x^q$.

- (d) If $q > 0$, then $x > y$ if and only if $x^q > y^q$.

- (e) If $x > 1$, then $x^q > x^r$ if and only if $q > r$. If $x < 1$, then $x^q > x^r$ if and only if $q < r$.

- **Proof**. See Week 5 homework. $\square$

- We still have to do real exponentiation; in other words, we still have to define x^y where $x > 0$ and y is a real number - but we will defer that for a little while.

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Some standard limits.

- Armed now with the limit laws and the squeeze test, we can now compute a large number of limits.

- A particularly simple limit is that of the *constant sequence* $c, c, c, c, \ldots$; we clearly have

$$\lim_{n\to\infty} c = c$$

for any constant c (why?).

- Also, in Proposition 17 from Week 3/4, we proved that $\lim_{n\to\infty} 1/n = 0$. This now implies

- **Corollary 5.** For any integer $k \geq 1$, we have $\lim_{n\to\infty} 1/n^{1/k} = 0$.

- **Proof.** From Lemma 2 we know that $1/n^{1/k}$ is a decreasing function of n, while being bounded below by 0. By Proposition 25 of Week 3/4 notes (for decreasing sequences instead of increasing sequences) we thus know that this sequence converges to some limit $L \geq 0$:

  $$L = \lim_{n\to\infty} 1/n^{1/k}.$$

  Raising this to the k^{th} power and using the limit laws (or more precisely, Theorem 21(b) of Week 3/4 notes and induction), we obtain

  $$L^k = \lim_{n\to\infty} 1/n.$$

  By Proposition 17 from Week 3/4 notes we thus have $L^k = 0$; but this means that L cannot be positive (else L^k would be positive), and thus $L = 0$, and we are done. $\square$.

- From Corollary 5 and the limit laws again we can conclude in fact that $\lim_{n\to\infty} 1/n^q = 0$ for any rational $q > 0$ (why?). This implies, by the way, that the limit $\lim_{n\to\infty} n^q$ does not exist (why? use Theorem 21(e) of Week 3/4 notes and prove by contradiction).

- From page 27 of Week 3/4 notes we also have the limit $\lim_{n\to\infty} x^n = 0$ when $0 < x < 1$. This also implies that $\lim_{n\to\infty} x^n = 0$ when $-1 < x < 0$ (why? Use Q10 of Week 4 Homework), and that $\lim_{n\to\infty} x^n$ does not exist when $x > 1$ or $x < -1$ (why? Use Theorem 21(e) of Week 3/4 notes and argue by contradiction). Quiz: what happens in the remaining cases $x = -1, 0, 1$?

- Another basic limit is the following:

- **Lemma 6.** For any $x > 0$, we have $\lim_{n\to\infty} x^{1/n} = 1$.

- **Proof.** See Week 5 homework. $\square$

- We will derive a few more standard limits later on, once we develop the root and ratio tests for series and for sequences.

- Finally, from Proposition 18 and Theorem 30 from last week's notes we recall that a Cauchy sequence of real numbers and a convergent sequence of real numbers are the same thing:

- **Proposition 7.** Let $(a_n)_{n=m}^\infty$ be a sequence of real numbers. Then $(a_n)_{n=m}^\infty$ is convergent if and only if it is a Cauchy sequence.

- **Proof.** See Proposition 18 and Theorem 30 from last week's notes. $\square$

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Finite series

- Now that we have developed a reasonable theory of limits of sequences, we will use that theory to develop a theory of infinite series

$$\sum_{n=m}^\infty a_n = a_m + a_{m+1} + a_{m+2} + \ldots.$$

  But before we develop infinite series, we must first develop the theory of finite series.

- Let m, n be integers, and let $(a_i)_{i=m}^n$ be a finite sequence of real numbers, assigning a real number a_i to each integer i between m and n inclusive (i.e. $m \leq i \leq n$). Then we define the finite sum (or finite series) $\sum_{i=m}^n a_i$ by the recursive formula

$$\sum_{i=m}^n a_i := 0 \text{ whenever } n < m; \sum_{i=m}^{n++} a_i := (\sum_{i=m}^n a_i) + a_{n++} \text{ whenever } n \geq m-1.$$

- Thus for instance

$$\sum_{i=m}^{m-2} a_i = 0; \quad \sum_{i=m}^{m-1} a_i = 0; \quad \sum_{i=m}^{m} a_i = a_m;$$

$$\sum_{i=m}^{m+1} a_i = a_m + a_{m+1}; \quad \sum_{i=m}^{m+2} a_i = a_m + a_{m+1} + a_{m+2}$$

(why do these identities follow from the above definition?). Because of this, we sometimes express $\sum_{i=m}^n a_i$ less formally as

$$\sum_{i=m}^n a_i = a_m + a_{m+1} + \ldots + a_n.$$

- (A remark: the difference between "sum" and "series" is a subtle linguistic one. Strictly speaking, a series is an *expression* of the form $\sum_{i=m}^n a_i$; this series is equal to a real number, which is then the *sum* of that series. For instance, $1+2+3+4+5$ is a series, whose sum is 15; if one were to be very picky about semantics, one would not consider 15 a series and one would not consider $1+2+3+4+5$ a sum, despite the two expressions having the same value. However, we will not be very careful about this distinction as it is purely linguistic and has no bearing on the mathematics; from the point of view of mathematics $1+2+3+4+5$ and 15 are the same number, and thus interchangeable (the axiom of substitution)).

- Note that the variable i (sometimes called the *index of summation*) is a *bound variable* (sometimes called a *dummy variable*); the expression $\sum_{i=m}^n a_i$ does not actually depend on any quantity named i. In particular, one can replace the index of summation i with any other symbol, and obtain the same sum:

$$\sum_{i=m}^n a_i = \sum_{j=m}^n a_j.$$

- We list some basic properties of summation below.

- **Lemma 8**

- (a) Let $m \leq n < p$ be integers, and let a_i be a real number assigned to each integer $m \leq i \leq p$. Then we have

$$\sum_{i=m}^n a_i + \sum_{i=n+1}^p a_i = \sum_{i=m}^p a_i.$$

- (b) Let $m \leq n$ be integers, k be another integer, and let a_i be a real number assigned to each integer $m \leq i \leq n$. Then we have

$$\sum_{i=m}^{n} a_i = \sum_{j=m+k}^{n+k} a_{j-k}.$$

- (c) Let $m \leq n$ be integers, and let a_i, b_i be real numbers assigned to each integer $m \leq i \leq n$. Then we have

$$\sum_{i=m}^{n}(a_i + b_i) = (\sum_{i=m}^{n} a_i) + (\sum_{i=m}^{n} b_i).$$

- (d) Let $m \leq n$ be integers, and let a_i be a real number assigned to each integer $m \leq i \leq n$, and let c be another real number. Then we have

$$\sum_{i=m}^{n}(ca_i) = c(\sum_{i=m}^{n} a_i).$$

- (e) (Triangle inequality for finite series) Let $m \leq n$ be integers, and let a_i be a real number assigned to each integer $m \leq i \leq n$. Then we have

$$|\sum_{i=m}^{n} a_i| \leq \sum_{i=m}^{n} |a_i|.$$

- (f) (Comparison test for finite series) Let $m \leq n$ be integers, and let a_i, b_i be real numbers assigned to each integer $m \leq i \leq n$. Suppose that $a_i \leq b_i$ for all $m \leq i \leq n$. Then we have

$$\sum_{i=m}^{n} a_i \leq \sum_{i=m}^{n} b_i.$$

- **Proof.** See Week 5 homework. □

- Note: in the future we may omit some of the parentheses in series expressions, for instance we may write $\sum_{i=m}^{n}(a_i + b_i)$ simply as $\sum_{i=m}^{n} a_i + b_i$. This is reasonably safe from being mis-interpreted, because the alternative interpretation $(\sum_{i=m}^{n} a_i) + b_i$ does not make any sense (the index i in b_i is meaningless outside of the summation, since i is only a dummy variable).

- One can use finite series to also define summations over finite sets:

- **Definition.** Let X be a finite set with n elements (where $n \in \mathbf{N}$), and let $f : X \to \mathbf{R}$ be a function from X to the real numbers (i.e. f assigns a real number $f(x)$ to each element x of X). Then we can define the finite sum $\sum_{x \in X} f(x)$ as follows. We first select any bijection g from $\{i \in \mathbf{N} : 1 \leq i \leq n\}$ to X; such a bijection exists since X is assumed to have n elements. We then define

$$\sum_{x \in X} f(x) := \sum_{i=1}^{n} f(g(i)).$$

- **Example** Let X be the three-element set $X := \{a, b, c\}$, where a, b, c are distinct objects, and let $f : X \to \mathbf{R}$ be the function $f(a) := 2$, $f(b) := 5$, $f(c) := -1$. In order to compute the sum $\sum_{x \in X} f(x)$, we select a bijection $g : \{1, 2, 3\} \to X$, e.g. $g(1) := a$, $g(2) := b$, $g(3) := c$. We then have

$$\sum_{x \in X} f(x) = \sum_{i=1}^{3} f(g(i)) = f(a) + f(b) + f(c) = 6.$$

  One could pick another bijection from $\{1, 2, 3\}$ to X, e.g. $h(1) := c$, $h(2) := b$, $h(3) = c$, but the end result is still the same:

$$\sum_{x \in X} f(x) = \sum_{i=1}^{3} f(h(i)) = f(c) + f(b) + f(a) = 6.$$

- To verify that this definition actually does give a single, well-defined value to $\sum_{x \in X} f(x)$, one has to check that different bijections g from $\{i \in \mathbf{N} : 1 \leq i \leq n\}$ to X give the same sum. In other words, we must prove

- **Proposition 9.** Let X be a finite set with n elements (where $n \in \mathbf{N}$), let $f : X \to \mathbf{R}$ be a function, and let $g : \{i \in \mathbf{N} : 1 \leq i \leq n\} \to X$ and $h : \{i \in \mathbf{N} : 1 \leq i \leq n\} \to X$ be bijections. Then we have

$$\sum_{i=1}^{n} f(g(i)) = \sum_{i=1}^{n} f(h(i)).$$

- **Proof.** We use induction on n; more precisely, we let $P(n)$ be the assertion that "For any set X of n elements, any function $f : X \to \mathbf{R}$, and any two bijections g, h from $\{i \in \mathbf{N} : 1 \leq i \leq n\}$ to X, we have $\sum_{i=1}^n f(g(i)) = \sum_{i=1}^n f(h(i))$". (More informally, $P(n)$ is the assertion that Proposition 9 is true for that value of n). We want to prove that $P(n)$ is true for all natural numbers n.

  We first check the base case $P(0)$. In this case we have $\sum_{i=1}^0 f(g(i))$ and $\sum_{i=1}^0 f(h(i))$ both equal to 0, by definition of finite series, so we are done.

  Now suppose inductively that $P(n)$ is true; we now prove that $P(n{+}{+})$ is true. Thus, let X be a set with $n++$ elements, let $f : X \to \mathbf{R}$ be a function, and let g and h be bijections from $\{i \in \mathbf{N} : 1 \leq i \leq n++\}$ to X. We have to prove that

  $$\sum_{i=1}^{n++} f(g(i)) = \sum_{i=1}^{n++} f(h(i)). \qquad (1)$$

  Let $x := g(n{+}{+})$; thus x is an element of X. By definition of finite series, we can expand the left-hand side of (1) as

  $$\sum_{i=1}^{n++} f(g(i)) = (\sum_{i=1}^{n} f(g(i))) + x.$$

  Now let us look at the right-hand side of (1). Ideally we would like to have $h(n{+}{+})$ also equal to x - this would allow us to use the inductive hypothesis $P(n)$ much more easily - but we cannot assume this. However, since h is a bijection, we do know that there is *some* index j, with $1 \leq j \leq n++$, for which $h(j) = x$. We now use Lemma 8 and the definition of finite series to write

  $$\sum_{i=1}^{n++} f(h(i)) = (\sum_{i=1}^{j} f(h(i))) + (\sum_{i=j+1}^{n+1} f(h(i)))$$

  $$= (\sum_{i=1}^{j-1} f(h(i))) + f(h(j)) + (\sum_{i=j+1}^{n+1} f(h(i)))$$

$$= (\sum_{i=1}^{j-1} f(h(i))) + x + (\sum_{i=j}^{n} f(h(i+1))).$$

We now define the function $\tilde{h} : \{i \in \mathbf{N} : 1 \leq i \leq n\} \to X - \{x\}$ by setting $\tilde{h}(i) := h(i)$ when $i < j$ and $\tilde{h}(i) := h(i+1)$ when $i \geq j$. We can thus write the right-hand side of (1) as

$$= (\sum_{i=1}^{j-1} f(\tilde{h}(i))) + x + (\sum_{i=j}^{n} f(\tilde{h}(i)))$$

$$= (\sum_{i=1}^{n} f(\tilde{h}(i))) + x$$

where we have used Lemma 8 once again. Thus to finish the proof of (1) we have to show that

$$\sum_{i=1}^{n} f(g(i)) = \sum_{i=1}^{n} f(\tilde{h}(i)). \qquad (2)$$

But the function g (when restricted to $\{i \in \mathbf{N} : 1 \leq i \leq n\}$) is a bijection from $\{i \in \mathbf{N} : 1 \leq i \leq n\} \to X - \{x\}$ (why?). The function $\tilde{h}$ is also a bijection from $\{i \in \mathbf{N} : 1 \leq i \leq n\} \to X - \{x\}$ (why? cf. Lemma 31 from Week 2 notes). Since $X - \{x\}$ has n elements (Lemma 31 from Week 2 notes), the claim (2) then follows directly from the induction hypothesis $P(n)$. □

- Because of Proposition 9, we know that our definition of $\sum_{x \in X} f(x)$ is well-defined.

- Some basic properties of summations on finite sets:

- **Proposition 10.**

- (a) If X is empty, and $f : X \to \mathbf{R}$ is a function (which, by the way, must be the *empty function*), we have

$$\sum_{x \in X} f(x) = 0.$$

- (b) If X consists of a single element, $X = \{x_0\}$, and $f : X \to \mathbf{R}$ is a function, we have
$$\sum_{x \in X} f(x) = f(x_0).$$

- (c) (Substitution rule) If X is a finite set, $f : X \to \mathbf{R}$ is a function, and $g : Y \to X$ is a bijection, then
$$\sum_{x \in X} f(x) = \sum_{y \in Y} f(g(y)).$$

- (d) Let $n \leq m$ be integers, and let X be the set $X := \{i \in \mathbf{Z} : n \leq i \leq m\}$. If a_i is a real number assigned to each integer $i \in X$, then we have
$$\sum_{i=n}^{m} a_i = \sum_{i \in X} a_i.$$

- (e) Let X, Y be disjoint finite sets (so $X \cap Y = \emptyset$), and $f : X \cup Y \to \mathbf{R}$ is a function. Then we have
$$\sum_{z \in X \cup Y} f(z) = (\sum_{x \in X} f(x)) + (\sum_{y \in Y} f(y)).$$

- (f) Let X be a finite set, and let $f : X \to \mathbf{R}$ and $g : X \to \mathbf{R}$ be functions. Then
$$\sum_{x \in X} f(x) + g(x) = (\sum_{x \in X} f(x)) + (\sum_{x \in X} g(x)).$$

- (g) Let X be a finite set, let $f : X \to \mathbf{R}$ be a function, and let c be a real number. Then
$$\sum_{x \in X} cf(x) = c \sum_{x \in X} f(x).$$

- (h) Let X be a finite set, and let $f : X \to \mathbf{R}$ and $g : X \to \mathbf{R}$ be functions such that $f(x) \leq g(x)$ for all $x \in X$. Then we have
$$\sum_{x \in X} f(x) \leq \sum_{x \in X} g(x).$$

- (i) Let X be a finite set, and let $f : X \to \mathbf{R}$ be a function, then

$$|\sum_{x \in X} f(x)| \leq \sum_{x \in X} |f(x)|.$$

- **Proof.** See Week 5 homework. $\square$

- The substitution rule (c) can be thought of as making the substitution $x := g(y)$ (hence the name). Note that the assumption that g is a bijection is essential; can you see why the rule will fail when g is not one-to-one or not onto? From (c) and (d) we see that

$$\sum_{i=n}^{m} a_i = \sum_{i=n}^{m} a_{f(i)}$$

  for any bijection f from the set $\{i \in \mathbf{Z} : n \leq i \leq m\}$ to itself. Informally, this means that we can rearrange the elements of a finite sequence at will and still obtain the same bound.

- Now we look at double finite series - finite series of finite series - and how they connect with Cartesian products.

- **Lemma 11.** Let X, Y be finite sets, and let $f : (X \times Y) \to \mathbf{R}$ be a function. Then

$$\sum_{x \in X} (\sum_{y \in Y} f(x,y)) = \sum_{(x,y) \in X \times Y} f(x,y).$$

- **Example.** Let $X := \{a, b\}$ and $Y := \{c, d\}$. Then the left-hand side is

$$\sum_{x \in \{a,b\}} (\sum_{y \in \{c,d\}} f(x,y)) = (f(a,c) + f(a,d)) + (f(b,c) + f(b,d))$$

  while the right-hand side is

$$\sum_{(x,y) \in \{(a,c),(b,c),(a,d),(b,d)\}} = f(a,c) + f(b,c) + f(a,d) + f(b,d)$$

  which is equal to the left-hand side.

- **Proof.** Let n be the number of elements in X. We will use induction on n (cf. Proposition 9); i.e. we let $P(n)$ be the assertion that Lemma 11 is true for any set X with n elements, and any finite set Y and any function $f : (X \times Y) \to \mathbf{R}$. We wish to prove $P(n)$ for all natural numbers n.

- The base case $P(0)$ is easy, following from Proposition 10(a) (why?). Now suppose that $P(n)$ is true; we now show that $P(n{+}{+})$ is true.

- Let X be a set with $n{+}{+}$ elements. In particular, by Lemma 31 from Week 2 notes, we can write $X = X' \cup \{x_0\}$, where x_0 is an element of X and $X' := X - \{x_0\}$ has n elements. Then by Proposition 10(e) we have

$$\sum_{x \in X} (\sum_{y \in Y} f(x,y)) = (\sum_{x \in X'} (\sum_{y \in Y} f(x,y))) + (\sum_{y \in Y} f(x_0, y));$$

by the induction hypothesis this is equal to

$$\sum_{(x,y) \in X' \times Y} f(x,y) + (\sum_{y \in Y} f(x_0, y)).$$

By Proposition 10(c) this is equal to

$$\sum_{(x,y) \in X' \times Y} f(x,y) + (\sum_{(x,y) \in \{x_0\} \times Y} f(x,y)).$$

By Proposition 10(e) this is equal to

$$\sum_{(x,y) \in X \times Y} f(x,y)$$

(why?) as desired. □

- **Corollary 12. (Fubini's theorem for finite sums)** Let X, Y be finite sets, and let $f : (X \times Y) \to \mathbf{R}$ be a function. Then

$$\sum_{x \in X} (\sum_{y \in Y} f(x,y)) = \sum_{(x,y) \in X \times Y} f(x,y) = \sum_{(y,x) \in Y \times X} f(x,y) = \sum_{y \in Y} (\sum_{x \in X} f(x,y)).$$

- **Proof.** In light of Lemma 11, it suffices to show that

$$\sum_{(x,y)\in X\times Y} f(x,y) = \sum_{(y,x)\in Y\times X} f(x,y).$$

But this follows from Proposition 10(c) by applying the bijection $h : X \times Y \to Y \times X$ defined by $h(x,y) := (y,x)$. (Why is this a bijection, and why does Proposition 10(c) give what we want?). $\square$

- This should be contrasted with the interchanging sums example on Page 4 of Week 1 notes; thus we anticipate something interesting to happen when we move from finite sums to infinite sums.

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Infinite series

- We are now ready to sum infinite series.

- **Definition.** A (formal) infinite series is any expression of the form

$$\sum_{n=m}^{\infty} a_n,$$

where m is an integer, and a_n is a real number for any integer $n \geq m$. We sometimes write this series as

$$a_m + a_{m+1} + a_{m+2} + \ldots.$$

- At present, this series is only defined *formally*; we have not set this sum equal to any real number; the notation $a_m + a_{m+1} + a_{m+2} + \ldots$ is of course designed to look very suggestively like a sum, but is not actually a finite sum because of the "..." symbol. To rigorously define what the series actually sums to, we need another definition.

- **Definition.** Let $\sum_{n=m}^{\infty} a_n$ be a formal infinite series. For any integer $N \geq m$, we define the N^{th} *partial sum* S_N of this series to be $S_N := \sum_{n=m}^{N} a_n$; of course, S_N is a real number. If the sequence $(S_N)_{n=m}^{\infty}$ converges to some limit L as $N \to \infty$, then we say that the infinite series $\sum_{n=m}^{\infty} a_n$ is *convergent*, and *converges to* L; we also write $L =$

$\sum_{n=m}^{\infty} a_n$, and say that L is the *sum* of the infinite series $\sum_{n=m}^{\infty} a_n$. If the partial sums S_N diverge, then we say that the infinite series $\sum_{n=m}^{\infty} a_n$ is *divergent*, and we do not assign any real number value to that series.

- **Examples.** Consider the formal infinite series

$$\sum_{n=1}^{\infty} 2^{-n} = 2^{-1} + 2^{-2} + 2^{-3} + \ldots.$$

The partial sums can be verified to equal

$$S_N = \sum_{n=1}^{N} 2^{-n} = 1 - 2^{-N}$$

by an easy induction argument (or by deriving the geometric series formula); the sequence $1 - 2^{-N}$ converges to 1 as $N \to \infty$, and hence we have

$$\sum_{n=1}^{\infty} 2^{-n} = 1.$$

In particular, this series is convergent. On the other hand, if we consider the series

$$\sum_{n=1}^{\infty} 2^{n} = 2^1 + 2^2 + 2^3 + \ldots$$

then the partial sums are

$$S_N = \sum_{n=1}^{N} 2^{n} = 2^{N+1} - 2$$

and this is easily shown to be an unbounded sequence, and hence divergent. Thus the series $\sum_{n=1}^{\infty} 2^n$ is divergent. Quiz: is the series $\sum_{n=1}^{\infty} (-1)^n$ convergent or divergent?

- Now we address the question of when a series converges. The following proposition shows that a series converges iff the "tail" of the sequence is eventually less than ε for any $\varepsilon > 0$:

- **Proposition 13.** Let $\sum_{n=m}^{\infty} a_n$ be a formal series of real numbers. Then $\sum_{n=m}^{\infty} a_n$ converges if and only if, for every real number $\varepsilon > 0$, there exists an integer $N \geq m$ such that

$$|\sum_{n=p}^{q} a_n| \leq \varepsilon \text{ for all } p, q \geq N.$$

- **Proof.** See Week 5 homework. $\square$

- This Proposition, by itself, is not very handy, because it is not so easy to compute the partial sums $\sum_{n=p}^{q} a_n$ in practice. However, it has a number of useful corollaries. For instance:

- **Corollary 14 (Zero test).** Let $\sum_{n=m}^{\infty} a_n$ be a formal series of real numbers. Then if $\sum_{n=m}^{\infty} a_n$ converges, we must have $\lim_{n\to\infty} a_n = 0$. To put this another way, if $\lim_{n\to\infty} a_n$ is non-zero or divergent, then the series $\sum_{n=m}^{\infty} a_n$ is divergent.

- **Proof.** See Week 5 homework. $\square$

- **Example.** The sequence $a_n := 1$ does not converge to 0 as $n \to \infty$, so we know that $\sum_{n=1}^{\infty} 1$ is a divergent series. (Note however that $1, 1, 1, 1, \ldots$ is a convergent *sequence*; convergence of series is a different notion from convergence of sequences). Similarly, the sequence $a_n := (-1)^n$ diverges, and in particular does not converge to zero; thus the series $\sum_{n=1}^{\infty} (-1)^n$ is also divergent.

- If a sequence $(a_n)_{n=m}^{\infty}$ *does* converge to zero, then the series $\sum_{n=m}^{\infty} a_n$ may or may not be convergent; it depends on the series. For instance, we will soon see that the series $\sum_{n=1}^{\infty} 1/n$ is divergent despite the fact that $1/n$ converges to 0 as $n \to \infty$.

- **Definition.** Let $\sum_{n=m}^{\infty} a_n$ be a formal series of real numbers. We say that this series is *absolutely convergent* iff the series $\sum_{n=m}^{\infty} |a_n|$ is convergent. If a series is not absolutely convergent, then it is absolutely divergent.

- In order to distinguish convergence from absolute convergence, we sometimes refer to the former as *conditional* convergence.

- **Proposition 15 (Absolute convergence test).** Let $\sum_{n=m}^{\infty} a_n$ be a formal series of real numbers. If this series is absolutely convergent, then it is also conditionally convergent. Furthermore, in this case we have the triangle inequality

$$|\sum_{n=m}^{\infty} a_n| \leq \sum_{n=m}^{\infty} |a_n|$$

- **Proof.** See Week 5 homework. □

- The converse, however, is not true; there exist series which are conditionally convergent but not absolutely convergent. We will give an example shortly.

- Note in particular that we consider the class of conditionally convergent series to include the class of absolutely convergent series as a subclass. Thus when we say a statement such as "$\sum_{n=m}^{\infty} a_n$ is conditionally convergent", this does not automatically mean that $\sum_{n=m}^{\infty}$ is not absolutely convergent. If we wish to say that a series is conditionally convergent but not absolutely convergent, then we will instead use a phrasing such as "$\sum_{n=m}^{\infty} a_n$ is only *conditionally* convergent", or "$\sum_{n=m}^{\infty} a_n$ converges conditionally, but not absolutely".

- **Proposition 16 (Alternating series test).** Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers which are non-negative and decreasing, thus $a_n \geq 0$ and $a_n \geq a_{n+1}$ for every $n \geq m$. Then the series $\sum_{n=m}^{\infty}(-1)^n a_n$ is convergent if and only if the sequence a_n converges to 0 as $n \to \infty$.

- **Proof.** From the zero test, we know that if the series $\sum_{n=m}^{\infty}(-1)^n a_n$ converges, then the sequence $(-1)^n a_n$ converges to 0, which implies (why? note that $(-1)^n a_n$ and a_n have the same distance from 0) that a_n also converges to 0.

- Now suppose conversely that a_n converges to 0. For each N, let S_N be the partial sum $S_N := \sum_{n=m}^{N}(-1)^n a_n$; our job is to show that S_N converges. Observe that

$$S_{N+2} = S_N + (-1)^{N+1} a_{N+1} + (-1)^{N+2} a_{N+2} = S_N + (-1)^{N+1}(a_{N+1} - a_{N+2}).$$

But by hypothesis, $(a_{N+1} - a_{N+2})$ is non-negative. Thus we have $S_{N+2} \geq S_N$ when N is odd and $S_{N+2} \leq S_N$ if N is even.

- Now suppose that N is even. From the above discussion and induction we see that $S_{N+2k} \leq S_N$ for all natural numbers k (why?). Also we have $S_{N+2k+1} \geq S_{N+1} = S_N - a_{N+1}$ (why?). Finally, we have $S_{N+2k+1} = S_{N+2k} - a_{N+2k+1} \leq S_{N+2k}$ (why?). Thus we have

$$S_N - a_{N+1} \leq S_{N+2k+1} \leq S_{N+2k} \leq S_N$$

for all k. In particular, we have

$$S_N - a_{N+1} \leq S_n \leq S_N \text{ for all } n \geq N$$

(why?). In particular, the sequence S_n is eventually a_{N+1}-steady. But the sequence a_N converges to 0 as $N \to \infty$, thus this implies that S_n is eventually ε-steady for every $\varepsilon > 0$ (why?). Thus S_n converges, and so the series $\sum_{n=m}^{\infty}(-1)^n a_n$ is convergent. □

- **Example.** The sequence $(1/n)_{n=1}^{\infty}$ is non-negative, decreasing, and converges to zero. Thus $\sum_{n=1}^{\infty}(-1)^n/n$ is convergent (but it is not absolutely convergent, because $\sum_{n=1}^{\infty} 1/n$ diverges, as we shall see below). Thus absolute divergence does not imply conditional divergence, even though absolute convergence implies conditional convergence.

- Some basic identities concerning convergent series are collected below.

- **Proposition 17**

- (a) If $\sum_{n=m}^{\infty} a_n$ is a series of real numbers converging to x, and $\sum_{n=m}^{\infty} b_n$ is a series of real numbers converging to y, then $\sum_{n=m}^{\infty}(a_n + b_n)$ is also a convergent series, and converges to $x + y$. In particular, we have

$$\sum_{n=m}^{\infty}(a_n + b_n) = \sum_{n=m}^{\infty} a_n + \sum_{n=m}^{\infty} b_n.$$

- (b) If $\sum_{n=m}^{\infty} a_n$ is a series of real numbers converging to x, and c is a real number, then $\sum_{n=m}^{\infty}(ca_n)$ is also a convergent series, and converges to cx. In particular, we have

$$\sum_{n=m}^{\infty}(ca_n) = c\sum_{n=m}^{\infty} a_n.$$

- (c) Let $\sum_{n=m}^\infty a_n$ be a series of real numbers, and let $k \geq 0$ be an integer. If one of the two series $\sum_{n=m}^\infty a_n$ and $\sum_{n=m+k}^\infty a_n$ are convergent, then the other one is also, and we have the identity

$$\sum_{n=m}^\infty a_n = \sum_{n=m}^{m+k-1} a_n + \sum_{n=m+k}^\infty a_n.$$

- (d) Let $\sum_{n=m}^\infty a_n$ be a series of real numbers converging to x, and let k be an integer. Then $\sum_{n=m+k}^\infty a_{n-k}$ also converges to x.

- **Proof.** See Week 5 homework.

- From Proposition 17(c) we see that the convergence of a series does not depend on the first few elements of the series (though of course those elements do influence which value the series converges to). Because of this, we will usually not pay much attention as to what the initial index m of the series is.

* * * * *

Sums of non-negative numbers

- Now we specialize to consider sums $\sum_{n=m}^\infty a_n$ where all the terms a_n are non-negative. This situation comes up, for instance, from the absolute convergence test, since the absolute value $|a_n|$ of a real number a_n is always non-negative. Note that when all the terms in a series are non-negative, there is no distinction between conditional convergence and absolute convergence.

- Suppose $\sum_{n=m}^\infty a_n$ is a series of non-negative numbers. Then the partial sums $S_N := \sum_{n=m}^N a_n$ are increasing, i.e. $S_{N+1} \geq S_N$ for all $N \geq n$ (Why?). From Proposition 25 and Corollary 20 of Week 3/4 notes, we thus see that the sequence $(S_N)_{n=m}^\infty$ is convergent if and only if it has an upper bound M. In other words, we have just shown

- **Proposition 18** Let $\sum_{n=m}^\infty a_n$ be a formal series of non-negative real numbers. Then this series is convergent if and only if there is a real number M such that

$$\sum_{n=m}^N a_n \leq M \text{ for all integers } N \geq m.$$

- A simple corollary of this is
- **Corollary 19 (Comparison test).** Let $\sum_{n=m}^{\infty} a_n$ and $\sum_{n=m}^{\infty} b_n$ be two formal series of real numbers, and suppose that $|a_n| \leq b_n$ for all $n \geq m$. Then if $\sum_{n=m}^{\infty} b_n$ is convergent, then $\sum_{n=m}^{\infty} a_n$ is absolutely convergent, and in fact

$$|\sum_{n=m}^{\infty} a_n| \leq \sum_{n=m}^{\infty} |a_n| \leq \sum_{n=m}^{\infty} b_n.$$

- **Proof.** See Week 5 homework. □
- We can also run the comparison test in the contrapositive: if we have $|a_n| \leq b_n$ for all $n \geq m$, and $\sum_{n=m}^{\infty} a_n$ is absolutely divergent, then $\sum_{n=m}^{\infty} b_n$ is conditionally divergent. (Why does this follow immediately from Corollary 19?)
- A useful series to use in the comparison test is the *geometric series*

$$\sum_{n=0}^{\infty} x^n,$$

where x is some real number. If $|x| \geq 1$ then this series diverges, thanks to the zero test (why?). Now suppose $|x| < 1$. The partial sums can easily be evaluated by the *geometric series formula*

$$\sum_{n=0}^{N} x^n = (1 - x^{N+1})/(1 - x);$$

this formula can be easily verified by induction (how?). If $|x| < 1$, then we have $\lim_{N\to\infty} x^N = 0$ by the discussion at the beginning of these notes, and so by the limit laws

$$\lim_{N\to\infty} (1 - x^{N+1})/(1 - x) = 1/(1 - x)$$

and hence $\sum_{n=0}^{\infty} x^n$ converges to $1/(1 - x)$ when $|x| < 1$. Indeed, this convergence is absolute (why?).

- We now give a useful criterion, known as the *Cauchy criterion*, to test whether a series of non-negative but decreasing terms is convergent.

- **Proposition 20 (Cauchy criterion).** Let $(a_n)_{n=1}^\infty$ be a decreasing sequence of non-negative real numbers (so $a_n \geq 0$ and $a_{n+1} \leq a_n$ for all $n \geq 1$). Then the series $\sum_{n=1}^\infty a_n$ is convergent if and only if the series

$$\sum_{k=0}^{\infty} 2^k a_{2^k} = a_1 + 2a_2 + 4a_4 + 8a_8 + \ldots$$

  is convergent.

- It is interesting that this criterion only uses a small number of elements of the sequence a_n (namely, those elements whose index n is a power of 2, $n = 2^k$) in order to determine whether the whole sequence is convergent or not.

- **Proof.** Let $S_N := \sum_{n=1}^N a_n$ be the partial sums of $\sum_{n=1}^\infty a_n$, and let $T_K := \sum_{k=0}^K 2^k a_{2^k}$ be the partial sums of $\sum_{k=0}^K 2^k a_{2^k}$. In light of Proposition 18, our task is to show that the sequence $(S_N)_{N=1}^\infty$ is bounded if and only if the sequence $(T_K)_{K=0}^\infty$ is bounded. To do this we need the following claim:

- **Claim.** For any natural number K, we have

$$S_{2^{K+1}-1} \leq T_K \leq 2S_{2^K}.$$

- **Proof.** We use induction on K. First we prove the claim when $K = 0$, i.e.

$$S_1 \leq T_0 \leq 2S_1.$$

  This becomes

$$a_1 \leq a_1 \leq 2a_1$$

  which is clearly true, since a_1 is non-negative.

- Now suppose the claim has been proven for K, and now we try to prove it for $K + 1$:

$$S_{2^{K+2}-1} \leq T_{K+1} \leq 2S_{2^{K+1}}.$$

Clearly we have

$$T_{K+1} = T_K + 2^{K+1} a_{2^{K+1}}.$$

Also, we have (using Lemma 8(af) and the hypothesis that the a_n are decreasing)

$$S_{2^{K+1}} = S_{2^K} + \sum_{n=2^K+1}^{2^{K+1}} a_n \geq S_{2^K} + \sum_{n=2^K+1}^{2^{K+1}} a_{2^{K+1}} = S_{2^K} + 2^K a_{2^{K+1}}$$

and hence

$$2S_{2^{K+1}} \geq 2S_{2^K} + 2^{K+1} a_{2^{K+1}}.$$

Similarly we have

$$S_{2^{K+2}-1} = S_{2^{K+1}-1} + \sum_{n=2^{K+1}}^{2^{K+2}-1} a_n \leq S_{2^{K+1}-1} + \sum_{n=2^{K+1}}^{2^{K+2}-1} a_{2^{K+1}} = S_{2^{K+1}-1} + 2^{K+1} a_{2^{K+1}}.$$

Combining these inequalities with the induction hypothesis

$$S_{2^{K+1}-1} \leq T_K \leq 2S_{2^K}$$

we obtain

$$S_{2^{K+2}-1} \leq T_{K+1} \leq 2S_{2^{K+1}}$$

as desired. This proves the claim. □

- From this claim we see that if $(S_N)_{N=1}^\infty$ is bounded, then $(S_{2^K})_{K=0}^\infty$ is bounded, and hence $(T_K)_{K=0}^\infty$ is bounded. Conversely, if $(T_K)_{K=0}^\infty$ is bounded, then the claim implies that $S_{2^{K+1}-1}$ is bounded, i.e. there is an M such that $S_{2^{K+1}-1} \leq M$ for all natural numbers K. But one can easily show (using induction) that $2^{K+1} - 1 \geq K + 1$, and hence that $S_{K+1} \leq M$ for all natural numbers K, hence $(S_N)_{N=1}^\infty$ is bounded. □

- **Corollary 21.** Let $q > 0$ be a rational number. Then the series $\sum_{n=1}^\infty 1/n^q$ is convergent when $q > 1$ and divergent when $q \leq 1$.

- **Proof.** The sequence $(1/n^q)_{n=1}^\infty$ is non-negative and decreasing (by Lemma 4(d)), and so the Cauchy criterion applies. Thus this series is convergent if and only if

$$\sum_{k=0}^\infty 2^k \frac{1}{(2^k)^q}$$

is convergent. But by the laws of exponentiation (Lemma 4) we can rewrite this as the geometric series

$$\sum_{k=0}^{\infty}(2^{1-q})^k.$$

As mentioned earlier, the geometric series $\sum_{k=0}^{\infty} x^k$ converges if and only if $|x| < 1$. Thus the series $\sum_{n=1}^{\infty} 1/n^q$ will converge if and only if $|2^{1-q}| < 1$, which happens if and only if $q > 1$ (why? Try proving it just using Lemma 4, and without using logarithms). $\square$

- In particular, the series $\sum_{n=1}^{\infty} 1/n$ (also known as the *harmonic series*) is divergent, as claimed earlier. However, the series $\sum_{n=1}^{\infty} 1/n^2$ is convergent.

- (A digression: The quantity $\sum_{n=1}^{\infty} 1/n^q$, when it converges, is called $\zeta(q)$, the *Riemann-zeta function of* q. This function is very important in number theory, and in particular in the distribution of the primes; there is a very famous unsolved problem regarding this function, called the *Riemann hypothesis*, but to discuss it further is far beyond the scope of this course. I will mention however that there is a US$ 1 million prize - and instant fame among all mathematicians - attached to the solution to this problem).

* * * * *

Rearrangement of series

- One feature of finite sums is that no matter how one rearranges the terms in a sequence, the total sum is the same. For instance,

$$a_1 + a_2 + a_3 + a_4 + a_5 = a_4 + a_3 + a_5 + a_1 + a_2.$$

A more rigorous statement of this, involving bijections, has already appeared earlier, just after Proposition 10.

- One can ask whether the same thing is true for infinite series. If all the terms are non-negative, the answer is yes:

- **Proposition 22.** Let $\sum_{n=0}^{\infty} a_n$ be a convergent series of non-negative real numbers, and let $f : \mathbf{N} \to \mathbf{N}$ be a bijection. Then $\sum_{m=0}^{\infty} a_{f(m)}$ is also convergent, and has the same sum:

$$\sum_{n=0}^{\infty} a_n = \sum_{m=0}^{\infty} a_{f(m)}.$$

- **Proof.** (Optional) We introduce the partial sums $S_N := \sum_{n=0}^{N} a_n$ and $T_M := \sum_{m=0}^{M} a_{f(m)}$. We know that the sequences $(S_N)_{n=0}^{\infty}$ and $(T_M)_{m=0}^{\infty}$ are increasing. Write $L := \sup(S_N)_{n=0}^{\infty}$ and $L' := \sup(T_M)_{m=0}^{\infty}$. By Proposition 25 from Week 3/4 notes we know that L is finite, and in fact $L = \sum_{n=0}^{\infty} a_n$; by Proposition 25 again we see that we will thus be done as soon as we can show that $L' = L$.

- Fix M, and let Y be the set $Y := \{m \in \mathbf{N} : m \leq M\}$. Note that f is a bijection between Y and $f(Y)$. By Proposition 10, we have

$$T_M = \sum_{m=0}^{M} a_{f(m)} = \sum_{m \in Y} a_{f(m)} = \sum_{n \in f(Y)} a_n.$$

The sequence $(f(m))_{m=0}^{M}$ is finite, hence bounded, i.e. there exists an N such that $f(m) \leq N$ for all $m \leq N$. In particular $f(Y)$ is a subset of $\{n \in \mathbf{N} : n \leq N\}$, and so by Proposition 10 again (and the assumption that all the a_n are non-negative)

$$T_M = \sum_{n \in f(Y)} a_n \leq \sum_{n \in \{n \in \mathbf{N} : n \leq N\}} a_n = \sum_{n=0}^{N} a_n = S_N.$$

But since $(S_N)_{N=0}^{\infty}$ has a supremum of L, we thus see that $S_N \leq L$, and hence that $T_M \leq L$ for all M. Since L' is the least upper bound of $(T_M)_{M=0}^{\infty}$, this implies that $L' \leq L$.

- A very similar argument (using the inverse f^{-1} instead of f) shows that every S_N is bounded above by L', and hence $L \leq L'$. Combining these two inequalities we obtain $L = L'$, as desired. $\square$

- **Example** We know that the series

$$\sum_{n=1}^{\infty} 1/n^2 = 1 + 1/4 + 1/9 + 1/16 + 1/25 + 1/36 + \dots$$

is convergent. Thus, if we interchange every pair of terms, to obtain

$$1/4 + 1 + 1/16 + 1/9 + 1/36 + 1/25 + \dots$$

we know that this series is also convergent, and has the same sum.

- Now we ask what happens when the series is not non-negative. Then as long as the series is *absolutely* convergent, we can still do rearrangements:

- **Proposition 23.** Let $\sum_{n=0}^{\infty} a_n$ be an absolutely convergent series of real numbers, and let $f : \mathbf{N} \to \mathbf{N}$ be a bijection. Then $\sum_{m=0}^{\infty} a_{f(m)}$ is also absolutely convergent, and has the same sum:

$$\sum_{n=0}^{\infty} a_n = \sum_{m=0}^{\infty} a_{f(m)}.$$

- **Proof.** (Optional) We apply Proposition 22 to the infinite series $\sum_{n=0}^{\infty} |a_n|$, which by hypothesis is a convergent series of non-negative numbers. If we write $L := \sum_{n=0}^{\infty} |a_n|$, then by Proposition 22 we know that $\sum_{m=0}^{\infty} |a_{f(m)}|$ also converges to L.

- Now write $L' := \sum_{n=0}^{\infty} a_n$. We have to show that $\sum_{m=0}^{\infty} a_{f(m)}$ also converges to L'. In other words, given any $\varepsilon > 0$, we have to find an M such that $\sum_{m=0}^{M'} a_{f(m)}$ is ε-close to L' for every $M' \geq M$.

- To prove this, we use the fact that $\sum_{n=0}^{\infty} |a_n|$ converges to L. In particular, by Proposition 13 we can find an N such that $\sum_{n=p}^{q} |a_n| \leq \varepsilon/2$ for all $p, q \geq N$. In particular, by the triangle inequality we have $|\sum_{n=p}^{q} a_n| \leq \varepsilon/2$ for all $p, q \geq N$, implies that $|\sum_{n=N+1}^{\infty} a_n| \leq \varepsilon/2$ (why?), so by Proposition 17(c) we have that $\sum_{n=0}^{N} a_n$ is $\varepsilon/2$-close to L'.

- Now the sequence $(f^{-1}(n))_{n=0}^N$ is finite, hence bounded, so there exists an M such that $f^{-1}(n) \leq M$ for all $0 \leq n \leq N$. In particular, for any $M' \geq M$, the set $\{f(m) : m \in \mathbf{N}; m \leq M'\}$ contains $\{n \in \mathbf{N} : n \leq N\}$(why?). So by Proposition 10, for any $M' \geq M$

$$\sum_{m=0}^{M'} a_{f(m)} = \sum_{n \in \{f(m) : m \in \mathbf{N}; m \leq M'\}} a_n = \sum_{n=0}^{N} a_n + \sum_{n \in X} a_n$$

where X is the set

$$X = \{f(m) : m \in \mathbf{N}; m \leq M'\} - \{n \in \mathbf{N} : n \leq N\}.$$

The set X is finite, and is therefore bounded by some natural number q; we must therefore have

$$X \subseteq \{n \in \mathbf{N} : N + 1 \leq n \leq q\}$$

(why?). Thus

$$|\sum_{n \in X} a_n| \leq \sum_{n \in X} |a_n| \leq \sum_{n=N+1}^{q} |a_n| \leq \varepsilon/2$$

by our choice of N. Thus $\sum_{m=0}^{M'} a_{f(m)}$ is $\varepsilon/2$-close to $\sum_{n=0}^{N} a_n$, which as mentioned before is $\varepsilon/2$-close to L'. Thus $\sum_{m=0}^{M'} a_{f(m)}$ is ε-close to L for all $M' \geq M$, as desired. □

- Surprisingly, when the series is not absolutely convergent, then the rearrangements are very badly behaved. As an example, consider the series

$$1/3 - 1/4 + 1/5 - 1/6 + 1/7 - 1/8 + \ldots.$$

This series is not absolutely convergent (why?), but is conditionally convergent by the alternating series test, and in fact the sum can be seen to converge to a positive number (in fact, it converges to $\ln(2) - 1/2 = 0.193147\ldots$). Basically, the reason why the sum is positive is because the quantities $(1/3-1/4)$, $(1/5-1/6)$, $(1/7-1/8)$ are all positive, which can then be used to show that every partial sum is positive (why? you have to break into two cases, depending on whether there are an even or odd number of terms in the partial sum).

- If, however, we rearrange the series to have two negative terms to each positive term, thus

$$1/3 - 1/4 - 1/6 + 1/5 - 1/8 - 1/10 + 1/7 - 1/12 - 1/14 + \ldots$$

then the partial sums quickly become negative (this is because $(1/3 - 1/4 - 1/6)$, $(1/5 - 1/8 - 1/9)$, and more generally $(1/(2n+1) - 1/4n - 1/(4n+2))$ are all negative), and so this series in fact converges to a negative quantity (in fact, it converges to $(\ln(2)-1)/2 = -.153426\ldots$). There is in fact a surprising result of Riemann, which shows that an absolutely divergent series can in fact be rearranged to converge to *any* value (or rearranged to diverge, in fact!):

- **Theorem 24.** Let $\sum_{n=0}^\infty a_n$ be a series which is not absolutely convergent, and let L be any real number. Then there exists a bijection $f : \mathbf{N} \to \mathbf{N}$ such that $\sum_{m=0}^\infty a_{f(m)}$ converges conditionally to L.

- **Proof.** (Optional) See Theorem 3.54 of the textbook. $\square$

- To summarize, rearranging series is OK when the series is absolutely convergent, but is somewhat dangerous otherwise. (This is not to say that rearranging an absolutely divergent series necessarily gives you the wrong answer - in fact, physicists are often do maneuvers like this, and still (usually) obtain a correct answer at the end - but doing so is risky, unless it is backed by a rigorous result such as Proposition 23).

- In light of Proposition 23, we can now talk about sums on *countable sets*, provided that the sum is absolutely convergent.

- **Definition** Let X be a countable set, and let $f : X \to \mathbf{R}$ be a function. We say that the series $\sum_{x \in X} f(x)$ is *absolutely convergent* iff for some bijection $g : \mathbf{N} \to X$, the sum $\sum_{n=0}^\infty f(g(n))$ is absolutely convergent. We then define the sum of $\sum_{x \in X} f(x)$ by the formula

$$\sum_{x \in X} f(x) = \sum_{n=0}^\infty f(g(n)).$$

- From Proposition 23 (and Proposition 29 from Week 2 notes), one can show that these definitions do not depend on the choice of g, and so are well defined.

- We can now use this to derive an important theorem about double summations.

- **Theorem 25. (Fubini's theorem for infinite sums)** Let $f : \mathbf{N} \times \mathbf{N} \to \mathbf{R}$ be a function such that $\sum_{(n,m) \in \mathbf{N} \times \mathbf{N}} f(n,m)$ is absolutely convergent. Then we have

$$\sum_{n=0}^{\infty} (\sum_{m=0}^{\infty} f(n,m)) = \sum_{(n,m) \in \mathbf{N} \times \mathbf{N}} f(n,m) = \sum_{(m,n) \in \mathbf{N} \times \mathbf{N}} f(n,m) = \sum_{m=0}^{\infty} (\sum_{n=0}^{\infty} f(n,m)).$$

- In other words, we can switch the order of infinite sums *provided that the entire sum is absolutely convergent.* You should go back and compare this with Page 4 of the Week 1 notes!

- **Proof.** (A sketch only; this proof is considerably more complex than the other proofs, and is optional reading) The second equality follows easily from Proposition 23 (and Proposition 29 in Week 2 notes). We shall just prove the first equality, as the third is very similar (basically one switches the role of n and m).

  Let us first consider the case when $f(n,m)$ is always non-negative (we will deal with the general case later). Write $L := \sum_{(n,m) \in \mathbf{N} \times \mathbf{N}} f(n,m)$; our task is to show that $\sum_{n=0}^{\infty} (\sum_{m=0}^{\infty} f(n,m))$ converges to L.

  One can easily show that $\sum_{(n,m) \in X} f(n,m) \leq L$ for all finite sets X (Why? Use a bijection g between $\mathbf{N} \times \mathbf{N}$ and $\mathbf{N}$, and then use the fact that $g(X)$ is finite, hence bounded). In particular, for every $n \in \mathbf{N}$ and $M \in \mathbf{N}$ we have $\sum_{m=0}^{M} f(n,m) \leq L$, which implies by Proposition 25 of Week 3/4 notes that $\sum_{m=0}^{\infty} f(n,m)$ is convergent for each m. Similarly, for any $N \in \mathbf{N}$ and $M \in \mathbf{N}$ we have (by Corollary 12)

$$\sum_{n=0}^{N} \sum_{m=0}^{M} f(n,m) \leq \sum_{(n,m) \in X} f(n,m) \leq L$$

  where X is the set $\{(n,m) \in \mathbf{N} \times \mathbf{N} : n \leq N, m \leq M\}$ which is finite by Q8 of Assignment 3. Taking suprema of this as $M \to \infty$ we have (by limit laws, and an induction on N)

$$\sum_{n=0}^{N} \sum_{m=0}^{\infty} f(n,m) \leq L.$$

By Proposition 25 of Week 3/4 notes, this implies that $\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} f(n,m)$ converges, and

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} f(n,m) \le L.$$

To finish the proof, it will suffice to show that

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} f(n,m) \ge L-\varepsilon$$

for every $\varepsilon > 0$ (why will this be enough? Prove by contradiction). So, let $\varepsilon > 0$. By definition of L, we can then find a finite set $X \subseteq \mathbf{N}\times\mathbf{N}$ such that $\sum_{(n,m)\in X} f(n,m) \ge L-\varepsilon$ (why?). This set, being finite, must be contained in some set of the form $Y := \{(n,m) \in \mathbf{N}\times\mathbf{N} : n \le N; m \le M\}$ (why? use induction), thus by Corollary 12

$$\sum_{n=0}^{N}\sum_{m=0}^{M} f(n,m) = \sum_{(n,m)\in Y} f(n,m) \ge \sum_{(n,m)\in X} f(n,m) \ge L-\varepsilon$$

and hence

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} f(n,m) \ge \sum_{n=0}^{N}\sum_{m=0}^{\infty} f(n,m) \ge \sum_{n=0}^{N}\sum_{m=0}^{M} f(n,m) \ge L-\varepsilon$$

as desired.

- This proves the claim when the $f(n,m)$ are all non-negative. A similar argument works when the $f(n,m)$ are all non-positive (in fact, one can simply apply the result just obtained to the function $-f(n,m)$, and then use limit laws to remove the $-$. For the general case, note that any function $f(n,m)$ can be written (why?) as $f_+(n,m)+f_-(n,m)$, where $f_+(n,m)$ is the positive part of $f(n,m)$ (i.e. it equals $f(n,m)$ when $f(n,m)$ is positive, and 0 otherwise, and f_- is the negative part of $f(n,m)$ (it equals $f(n,m)$ when $f(n,m)$ is negative, and 0 otherwise. It is easy to show that if $\sum_{(n,m)\in\mathbf{N}\times\mathbf{N}} f(n,m)$ is absolutely convergent, then so is $\sum_{(n,m)\in\mathbf{N}\times\mathbf{N}} f_+(n,m)$ and $\sum_{(n,m)\in\mathbf{N}\times\mathbf{N}} f_-(n,m)$. So now one applies the results just obtained to f_+ and to f_- and adds them together using limit laws to obtain the result for general f. $\square$

* * * * *

The root and ratio tests.

- Now we can state and prove the famous root and ratio tests for convergence.

- **Theorem 26 (Root test).** Let $\sum_{n=m}^{\infty} a_n$ be a series of real numbers, and let $\alpha := \limsup_{n\to\infty} |a_n|^{1/n}$.

- (a) If $\alpha < 1$, then the series $\sum_{n=m}^{\infty} a_n$ is absolutely convergent (and hence conditionally convergent).

- (b) If $\alpha > 1$, then the series $\sum_{n=m}^{\infty} a_n$ is conditionally divergent (and hence absolutely divergent).

- (c) If $\alpha = 1$, we do not assert any conclusion.

- **Proof.** First suppose that $\alpha < 1$. Note that we must have $\alpha \geq 0$, since $|a_n|^{1/n} \geq 0$ for every n. Then we can find an $\varepsilon > 0$ such that $0 < \alpha + \varepsilon < 1$ (for instance, we can set $\varepsilon := (1-\alpha)/2$). By Proposition 27(a) of Week 3/4 notes, there exists an $N \geq m$ such that $|a_n|^{1/n} \leq \alpha+\varepsilon$ for all $n \geq N$. In other words, we have $|a_n| \leq (\alpha+\varepsilon)^n$ for all $n \geq N$. But from the geometric series we have that $\sum_{n=N}^{\infty}(\alpha+\varepsilon)^n$ is absolutely convergent, since $0 < \alpha + \varepsilon < 1$ (note that the fact that we start from N is irrelevant by Proposition 10(c)). Thus by the comparison test, we see that $\sum_{n=N}^{\infty} a_n$ is absolutely convergent, and thus $\sum_{n=m}^{\infty} a_n$ is absolutely convergent, by Proposition 10(c) again.

- Now suppose that $\alpha > 1$. Then by Proposition 27(b) of week 3/4 notes, we see that for every $N \geq m$ there exists an $n \geq N$ such that $|a_n|^{1/n} \geq 1$, and hence that $|a_n| \geq 1$. In particular, $(a_n)_{n=N}^{\infty}$ is not 1-close to 0 for any N, and hence $(a_n)_{n=m}^{\infty}$ is not eventually 1-close to 0. In particular, $(a_n)_{n=m}^{\infty}$ does not converge to zero. Thus by the zero test, $\sum_{n=m}^{\infty} a_n$ is conditionally divergent.

- When $\alpha = 1$, there is nothing to prove. $\square$

- As we shall see later, there are cases when $\alpha = 1$ when the series $\sum_{n=m}^{\infty} a_n$ converges, and there are other cases when $\alpha = 1$ but the series $\sum_{n=m}^{\infty} a_n$ diverges.

- The root test is phrased using the limit superior, but of course if $\lim_{n\to\infty} |a_n|^{1/n}$ converges then the limit is the same as the limit superior. Thus one can phrase the root test using the limit instead of the limit superior, but *only when the limit exists.*

- The root test is sometimes difficult to use; however we can replace roots by ratios using the following lemma.

- **Lemma 27.** Let $(c_n)_{n=m}^{\infty}$ be a sequence of positive numbers. Then we have

$$\liminf_{n\to\infty} \frac{c_{n+1}}{c_n} \leq \liminf_{n\to\infty} c_n^{1/n} \leq \limsup_{n\to\infty} c_n^{1/n} \leq \limsup_{n\to\infty} \frac{c_{n+1}}{c_n}.$$

- **Proof.** There are three inequalities to prove here. The middle inequality follows from Proposition 27(c) from Week 3/4 notes. We shall prove the last inequality, and leave the first one as homework.

- Write $L := \limsup_{n\to\infty} \frac{c_{n+1}}{c_n}$. If $L = +\infty$ then there is nothing to prove (since $x \leq +\infty$ for every extended real number x), so we may assume that L is a finite real number. (Note that L cannot equal $-\infty$; why?). Since $\frac{c_{n+1}}{c_n}$ is always positive, we know that $L \geq 0$.

- Let $\varepsilon > 0$. By Proposition 27(a) from Week 3/4 notes, we know that there exists an $N \geq m$ such that $\frac{c_{n+1}}{c_n} \leq L + \varepsilon$ for all $n \geq N$. This implies that $c_{n+1} \leq c_n(L+\varepsilon)$ for all $n \geq N$. By induction this implies that

$$c_n \leq c_N(L+\varepsilon)^{n-N} \text{ for all } n \geq N$$

(why?). If we write $A := c_N(L+\varepsilon)^{-N}$, then we have

$$c_n \leq A(L+\varepsilon)^n$$

and thus

$$c_n^{1/n} \leq A^{1/n}(L+\varepsilon)$$

for all $n \geq N$. But we have

$$\lim_{n\to\infty} A^{1/n}(L+\varepsilon) = L+\varepsilon$$

by the limit laws and Lemma 6. Thus by the comparison principle (Lemma 28 of Week 3/4 notes) we have

$$\limsup_{n\to\infty} c_n^{1/n} \leq L + \varepsilon.$$

But this is true for all $\varepsilon > 0$, so this must imply that

$$\limsup_{n\to\infty} c_n^{1/n} \leq L$$

(why? prove by contradiction), as desired. $\square$

- From Theorem 26 and Lemma 27 we immediately obtain
- **Corollary 28 (Ratio test).** Let $\sum_{n=m}^{\infty} a_n$ be a series of *non-zero* numbers. (The non-zero hypothesis is required so that the ratios $|a_{n+1}|/|a_n|$ are well-defined).
- (a) If $\limsup_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} < 1$, then the series $\sum_{n=m}^{\infty}$ is absolutely convergent (hence conditionally convergent).
- (b) If $\liminf_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} > 1$, then the series $\sum_{n=m}^{\infty}$ is conditionally divergent (hence absolutely divergent).
- Another consequence of Lemma 27 is the following limit:
- **Proposition 29.** We have $\lim_{n\to\infty} n^{1/n} = 1$.
- **Proof.** By Lemma 27 we have

$$\limsup_{n\to\infty} n^{1/n} \leq \limsup_{n\to\infty} (n+1)/n = \limsup_{n\to\infty} 1 + 1/n = 1$$

by Proposition 17 of Week 3/4 notes and limit laws. Similarly we have

$$\liminf_{n\to\infty} n^{1/n} \geq \liminf_{n\to\infty} (n+1)/n = \liminf_{n\to\infty} 1 + 1/n = 1.$$

The claim then follows from Proposition 27(ce) of Week 3/4 notes. $\square$

- From this Proposition, we see in particular that the series $\sum_{n=1}^{\infty} 1/n$ and $\sum_{n=1}^{\infty} 1/n^2$ both verify case (c) of the ratio test (Theorem 26). But we know from Corollary 21 that the first series is divergent, while the second series is absolutely convergent. Thus case (c) of the ratio test really is inconclusive.

* * * * *

Real exponentiation

- We finally return to the topic of exponentiation that we started at the beginning of this week's notes. We have already defined x^q for all rational q and positive real numbers x, but we have not yet defined x^α when α is real. We now rectify this situation using limits (in a similar way as to how we defined all the other standard operations on the real numbers). First, we need a lemma:

- **Lemma 30.** Let $x > 0$, and let α be a real number. Let $(q_n)_{n=1}^\infty$ be any sequence of rational numbers converging to α. Then $(x^{q_n})_{n=1}^\infty$ is also a convergent sequence. Furthermore, if $(q'_n)_{n=1}^\infty$ is any other sequence of rational numbers converging to α, then $(x^{q'_n})_{n=1}^\infty$ has the same limit as $(x^{q_n})_{n=1}^\infty$:
$$\lim_{n\to\infty} x^{q_n} = \lim_{n\to\infty} x^{q'_n}.$$

- **Proof.** There are three cases: $x < 1$, $x = 1$, and $x > 1$. The case $x = 1$ is rather easy (because then $x^q = 1$ for all rational q). We shall just do the case $x > 1$, and leave the case $x < 1$ (which is very similar) to the reader.

- Let us first prove that $(x^{q_n})_{n=1}^\infty$ converges. By Proposition 7 it is enough to show that $(x^{q_n})_{n=1}^\infty$ is a Cauchy sequence.

- To do this, we need to estimate the distance between x^{q_n} and x^{q_m}; let us say for the time being that $q_n \geq q_m$, so that $x^{q_n} \geq x^{q_m}$ (since $x > 1$). We have
$$d(x^{q_n}, x^{q_m}) = x^{q_n} - x^{q_m} = x^{q_m}(x^{q_n - q_m} - 1).$$
Since $(q_n)_{n=1}^\infty$ is a convergent sequence, it has some upper bound M; since $x > 1$, we have $x^{q_m} \leq x^M$. Thus
$$d(x^{q_n}, x^{q_m}) = |x^{q_n} - x^{q_m}| \leq x^M(x^{q_n - q_m} - 1).$$
Now let $\varepsilon > 0$. We know by Lemma 6 that the sequence $(x^{1/k})_{k=1}^\infty$ is eventually εx^{-M}-close to 1. Thus there exists some $K \geq 1$ such that
$$|x^{1/K} - 1| \leq \varepsilon x^{-M}.$$

Now since $(q_n)_{n=1}^\infty$ is convergent, it is a Cauchy sequence, and so there is an $N \geq 1$ such that q_n and q_m are $1/K$-close for all $n, m \geq N$. Thus we have

$$d(x^{q_n}, x^{q_m}) = x^M(x^{q_n - q_m} - 1) \leq x^M(x^{1/K} - 1) \leq x^M \varepsilon x^{-M} = \varepsilon$$

for every $n, m \geq N$ such that $q_n \geq q_m$. By symmetry we also have this bound when $n, m \geq N$ and $q_n \leq q_m$. Thus the sequence $(x^{q_n})_{n=N}^\infty$ is ε-steady. Thus the sequence $(x^{q_n})_{n=1}^\infty$ is eventually ε-steady for every $\varepsilon > 0$, and is thus a Cauchy sequence as desired. This proves the convergence of $(x^{q_n})_{n=1}^\infty$.

- Now we prove the second claim. It will suffice to show that

$$\lim_{n\to\infty} x^{q_n - q'_n} = 1,$$

since the claim would then follow from limit laws (since $x^{q_n} = x^{q_n - q'_n} x^{q'_n}$).

- Write $r_n := q_n - q'_n$; by limit laws we know that $(r_n)_{n=1}^\infty$ converges to 0. We have to show that for every $\varepsilon > 0$, the sequence $(x^{r_n})_{n=1}^\infty$ is eventually ε-close to 1. But from Lemma 6 we know that the sequence $(x^{1/k})_{k=1}^\infty$ is eventually ε-close to 1. Since $\lim_{k\to\infty} x^{-1/k}$ is also equal to 1 by Lemma 6, we also know that $(x^{-1/k})_{k=1}^\infty$ is also eventually ε-close to 1. Thus we can find a K such that $x^{1/K}$ and $x^{-1/K}$ are both ε-close to 1. But since $(r_n)_{n=1}^\infty$ is convergent to 0, it is eventually $1/K$-close to 0, so that eventually $-1/K \leq r_n \leq 1/K$, and thus $x^{-1/K} \leq x^{r_n} \leq x^{1/K}$. In particular x^{r_n} is also eventually ε-close to 1 (see Proposition 2(f) of Week 2 notes), as desired. $\square$

- We may now make the following definition.

- **Definition.** Let $x > 0$ be real, and let α be a real number. We define the quantity x^α by the formula $x^\alpha = \lim_{n\to\infty} x^{q_n}$, where $(q_n)_{n=1}^\infty$ is any sequence of rational numbers converging to α.

- Let us check that this definition is well-defined. First of all, given any real number α we always have at least one sequence $(q_n)_{n=1}^\infty$ of rational numbers converging to α, by the definition of real numbers (and Proposition 19 of Week 3/4 notes). Secondly, given any such sequence $(q_n)_{n=1}^\infty$, the limit $\lim_{n\to\infty} x^{q_n}$ exists by Lemma 30. Finally, even though there can be multiple choices for the sequence $(q_n)_{n=1}^\infty$, they all give the same limit by Lemma 30 again. Thus this definition is well-defined.

- If α is not just real but rational, i.e. $\alpha = q$ for some rational q, then this definition could in principle be inconsistent with our earlier definition of exponentiation. But in this case α is clearly the limit of the sequence $(q)_{n=1}^\infty$, so by definition $x^\alpha = \lim_{n\to\infty} x^q = x^q$. Thus the new definition of exponentiation is consistent with the old one.

- **Proposition 31.** All the results of Lemma 4, which held for rational numbers q and r, continue to hold for real numbers q and r.

- **Proof.** We demonstrate this for the identity $x^{q+r} = x^q x^r$ (i.e. the first part of Lemma 4(b)); the other parts are similar and are left to the reader. The idea is to start with Lemma 4 for rationals and then take limits to obtain Lemma 4 for reals.

- Let q and r be real numbers. Then we can write $q = \lim_{n\to\infty} q_n$ and $r = \lim_{n\to\infty} r_n$ for some sequences $(q_n)_{n=1}^\infty$ and $(r_n)_{n=1}^\infty$ of rationals, by the definition of real number (and Proposition 19 of Week 3/4 notes). Then by limit laws, $q + r$ is the limit of $(q_n + r_n)_{n=1}^\infty$. By definition of real exponentiation, we have

$$x^{q+r} = \lim_{n\to\infty} x^{q_n+r_n}; \quad x^q = \lim_{n\to\infty} x^{q_n}; \quad x^r = \lim_{n\to\infty} x^{r_n}.$$

But by Lemma 4(b) (applied to *rational* exponents) we have $x^{q_n+r_n} = x^{q_n} x^{r_n}$. Thus by limit laws we have $x^{q+r} = x^q x^r$, as desired. The other parts of Lemma 4 are proven similarly. $\square$